

CRYSTAL BASIS THEORY FOR A QUANTUM SYMMETRIC PAIR $(\mathbf{U}, \mathbf{U}^j)$

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ABSTRACT. We study the representation theory of a quantum symmetric pair $(\mathbf{U}, \mathbf{U}^j)$ with two parameters p, q of type AIII, by using highest weight theory and a variant of Kashiwara's crystal basis theory. Namely, we classify the irreducible \mathbf{U}^j -modules in a suitable category and associate with each of them a basis at $p = q = 0$, the j -crystal basis. The j -crystal basis of a finite-dimensional \mathbf{U} -module is thought of as a "localization" of the j -canonical basis, which was introduced by Huanchen Bao and Weiqiang Wang in 2013. Also, the j -crystal bases have nice combinatorial properties as the ordinary crystal bases do.

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1. INTRODUCTION

1.1. Quantum Schur-Weyl duality. Jimbo [J86] established a quantum analog of the classical Schur-Weyl duality. Let $U_q(\mathfrak{gl}_n)$ denote the quantum enveloping algebra of \mathfrak{gl}_n , and $\mathcal{H}(\mathfrak{S}_d)$ the Hecke algebra associated with the d -th symmetric group \mathfrak{S}_d , where $n \geq d$. Let V denote the vector representation of $U_q(\mathfrak{gl}_n)$. Jimbo defined an $\mathcal{H}(\mathfrak{S}_d)$ -module structure on $V^{\otimes d}$ by using the R -matrix for $V \otimes V$. Also, he proved that the actions of $U_q(\mathfrak{gl}_n)$ and $\mathcal{H}(\mathfrak{S}_d)$ on $V^{\otimes d}$ satisfy the double centralizer property, and hence, $V^{\otimes d}$ decomposes as a $U_q(\mathfrak{gl}_n)$ - $\mathcal{H}(\mathfrak{S}_d)$ -bimodule as:

$$V^{\otimes d} = \bigoplus_{\lambda \in \Lambda} V(\lambda) \boxtimes S^\lambda,$$

where Λ is an index set, and $\{V(\lambda) \mid \lambda \in \Lambda\}$ and $\{S^\lambda \mid \lambda \in \Lambda\}$ are families of nonisomorphic irreducible modules of $U_q(\mathfrak{gl}_n)$ and $\mathcal{H}(\mathfrak{S}_d)$, respectively.

1.2. Quantum Schur-Weyl duality in type B . It has been known that there is no Schur-Weyl-type duality between the quantum enveloping algebra of type B and the Hecke algebra $\mathcal{H}(W_d)$ of type B . However, Bao and Wang discovered the double centralizer property between a quantum symmetric pair and $\mathcal{H}(W_d)$ ([BW13]). More precisely, let $\mathbf{U}^J = \mathbf{U}_r^J$ be a coideal subalgebra of $\mathbf{U} = \mathbf{U}_{2r+1} = U_q(\mathfrak{sl}_{2r+1})$ such that $(\mathbf{U}, \mathbf{U}^J)$ forms a quantum analog of the symmetric pair of type AIII ([Le99], [Ko14]). In [BW13], Bao and Wang introduced the intertwiner Υ , which played a central role when they defined the action of $\mathcal{H}(W_d)$ on $V^{\otimes d}$, and then, proved that the actions of \mathbf{U}^J and $\mathcal{H}(W_d)$ on $V^{\otimes d}$ satisfy the double centralizer property. A variant of this work, where $\mathcal{H}(W_d)$ is replaced with the Hecke algebra of type B_d with unequal parameters (p, q) , was done in [BWW16].

1.3. Representation theory of \mathbf{U}^J . From the quantum Schur-Weyl duality in type B , we expect that there should exist a deep connection between the representation theory of \mathbf{U}^J and that of $\mathcal{H}(W_d)$. However, here arises a problem: although the representation theory of $\mathcal{H}(W_d)$ has been well-studied, little is known about that of \mathbf{U}^J . This paper gives some fundamental results in the representation theory of \mathbf{U}^J by using analogs of highest weight theory and Kashiwara's crystal basis theory.

In this paper, we treat the category $\mathcal{O}_{\text{int}}^J$ consisting of all \mathbf{U}^J -modules M satisfying the following: M is decomposed into its “weight spaces”, each of which is finite-dimensional; the set of weights of M is bounded from above; M is “integrable”.

We begin our study by decomposing \mathbf{U}^j into three parts. This is an analog of the triangular decomposition of \mathbf{U} . Using this triangular decomposition of \mathbf{U}^j , we define a “Verma module” associated with each weight. By its definition and the triangular decomposition of \mathbf{U}^j , it possesses a unique irreducible quotient. Our first main result is

Theorem A. *Every \mathbf{U}^j -module in $\mathcal{O}_{\text{int}}^j$ is completely reducible, and each irreducible \mathbf{U}^j -module in $\mathcal{O}_{\text{int}}^j$ is isomorphic to the irreducible quotient of a Verma module. Moreover, the isomorphism classes of irreducible \mathbf{U}^j -modules in $\mathcal{O}_{\text{int}}^j$ are parametrized by the pairs of partitions of length $r+1$ and r .*

Following Kashiwara’s crystal basis theory, we introduce the notions of j -crystal basis and its j -crystal graph; this can be thought of as a basis at $p = q = 0$. The second main result of this paper is

Theorem B. *Each irreducible \mathbf{U}^j -module in $\mathcal{O}_{\text{int}}^j$ admits a unique j -crystal basis whose j -crystal graph is connected with a single source.*

This theorem and the complete reducibility of \mathbf{U}^j -modules lead to the existence and uniqueness of j -crystal basis of a \mathbf{U}^j -module. Also, as in the ordinary crystal basis theory, j -crystal bases have the tensor product rule.

Theorem C. *Let M be a \mathbf{U}^j -module and N a \mathbf{U} -module. Suppose that M admits a j -crystal basis $(\mathcal{L}, \mathcal{B})$, and that N has a crystal basis $(\mathcal{L}', \mathcal{B}')$. Then, $(\mathcal{L} \otimes \mathcal{L}', \mathcal{B} \otimes \mathcal{B}')$ is a j -crystal basis of $M \otimes N$. In particular, (by taking M to be the trivial \mathbf{U}^j -module) the crystal basis of a \mathbf{U} -module N is the j -crystal basis of N .*

Here, let us recall a result in the representation theory of \mathbf{U}^j from [BW13]. In it, Bao and Wang introduced the notion of j -canonical basis for a finite-dimensional based \mathbf{U} -module (in the sense of [Lu94, Chapter 27]). They proved that a finite-dimensional based \mathbf{U} -module (M, B) admits a unique j -canonical basis $B^j := \{T_b \mid b \in B\}$ of the form

$$(1) \quad T_b = b + \sum_{b' \in B, b' \prec b} t_{b,b'} b', \quad t_{b,b'} \in q\mathbb{Z}[q],$$

where \prec denotes a partial order on B (see [BW13, Theorem 6.24] for details). By equation (1), the $\mathbb{Z}[q]$ -span of B^j coincides with that of B , and hence the set $\{T_b + qB^j \mid b \in B\}$ is the crystal basis of M . Thus, the j -crystal basis of M can be thought of as a “localization” of the j -canonical basis. Note that the category $\mathcal{O}_{\text{int}}^j$ contains objects other than finite-dimensional based \mathbf{U} -modules. For those objects, the notion of j -canonical basis has not been defined. We expect that we can “globalize” the j -crystal bases of such objects; namely, we expect that there exists a basis which we should call the j -canonical basis for each module in $\mathcal{O}_{\text{int}}^j$.

Finally, we mention that j -crystal bases have rich combinatorial properties. In particular, the j -crystal basis of an irreducible \mathbf{U}^j -module is realized as the set of pairs of semistandard Young tableaux of given shapes. As applications, we describe explicitly irreducible decompositions of $V_{2r+1}^{\otimes N}$ (Robinson-Shensted-type correspondence) and the tensor product of an irreducible \mathbf{U}^j -module with an irreducible \mathbf{U} -module (Littlewood-Richardson-type rule).

1.4. Organization of the paper. This paper is organized as follows. Section 2 is devoted to introducing the quantum enveloping algebra $\mathbf{U} = \mathbf{U}_{2r+1} = U_q(\mathfrak{sl}_{2r+1})$, its coideal subalgebra $\mathbf{U}^j = \mathbf{U}_r^j$, and the category $\mathcal{O}_{\text{int}}^j$.

We classify all the irreducible \mathbf{U}^j -modules in $\mathcal{O}_{\text{int}}^j$ and prove the complete reducibility of \mathbf{U}^j -modules in $\mathcal{O}_{\text{int}}^j$ for the case $r = 1$ in Section 3, and for a general r in Section 4.

In Section 5, we introduce the notion of quasi- j -crystal basis of an integrable \mathbf{U}_r^j -module in a naive way.

We study \mathbf{U}_2^j -modules in Section 6. We associate with each irreducible \mathbf{U}_2^j -module in $\mathcal{O}_{\text{int}}^j$ a quasi- j -crystal basis in a systematic way.

In Section 7, we define j -crystal bases by generalizing the quasi- j -crystal bases constructed in Section 6, and state our main result: the existence and uniqueness theorem for j -crystal bases of \mathbf{U}_r^j -modules in $\mathcal{O}_{\text{int}}^j$. Its proof is given in Section 9 since we need some combinatorial tools, which we prepare in Section 8.

We end this paper by giving some applications of j -crystal bases, such as Robinson-Shensted-type correspondence and Littlewood-Richardson-type rule, in Section 10.

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2. BASICS OF THE QUANTUM SYMMETRIC PAIR $(\mathbf{U}, \mathbf{U}^j)$

2.1. Definition of \mathbf{U}^j . Let $r \geq 1$, and set

$$\mathbb{I} := \left\{ -\left(r - \frac{1}{2}\right), -\left(r - \frac{3}{2}\right), \dots, r - \frac{1}{2} \right\}, \quad \mathbb{J} := \{1, 2, \dots, r\}.$$

Let Φ denote the root system of type A_{2r} with simple roots $\Pi = \{\alpha_i := \epsilon_{i-\frac{1}{2}} - \epsilon_{i+\frac{1}{2}} \mid i \in \mathbb{I}\}$, where $\{\epsilon_i \mid i = -r, -r+1, \dots, r\}$ is the standard basis of the Euclidean space \mathbb{R}^{2r+1} with the inner product (\cdot, \cdot) ; the associated Dynkin diagram is

$$\begin{array}{ccccccc} \bullet & \text{---} & \cdots & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \cdots & \text{---} & \bullet \\ -\left(r - \frac{1}{2}\right) & & & & -\frac{1}{2} & & \frac{1}{2} & & & & r - \frac{1}{2} \end{array}$$

We denote the set of positive roots by Φ_+ and the weight lattice by $\Lambda = \bigoplus_{i=-r}^r \mathbb{Z}\epsilon_i$.

Let $\mathbf{U} = \mathbf{U}_{2r+1}$ denote the quantum group $U_q(\mathfrak{sl}_{2r+1})$ of type A_{2r} over $\mathbb{Q}(p, q)$ (with p and q indeterminates) with generators E_i, F_i , and $K_i^{\pm 1}$, $i \in \mathbb{I}$, subject to the following relations:

$$\begin{aligned} K_i K_i^{-1} &= K_i^{-1} K_i = 1, \\ K_i K_j &= K_j K_i, \\ K_i E_j K_i^{-1} &= q^{(\alpha_i, \alpha_j)} E_j, \\ K_i F_j K_i^{-1} &= q^{-(\alpha_i, \alpha_j)} F_j, \\ E_i F_j - F_j E_i &= \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\ E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 &= 0 \quad \text{if } |i - j| = 1, \\ F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 &= 0 \quad \text{if } |i - j| = 1, \\ E_i E_j - E_j E_i &= 0 \quad \text{if } |i - j| > 1, \\ F_i F_j - F_j F_i &= 0 \quad \text{if } |i - j| > 1. \end{aligned}$$

Let \mathbf{U}^- denote the subalgebra of \mathbf{U} generated by F_i , $i \in \mathbb{I}$.

We employ the comultiplication Δ of \mathbf{U} given by:

$$\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \quad \Delta(E_i) = 1 \otimes E_i + E_i \otimes K_i^{-1}, \quad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i \quad \text{for } i \in \mathbb{I}.$$

Let $(\mathbf{U}, \mathbf{U}^j)$ denote the quantum symmetric pair over $\mathbb{Q}(p, q)$ of type AIII, that is, \mathbf{U}^j is the subalgebra of \mathbf{U} generated by

$$\begin{aligned} k_i^{\pm 1} &:= (K_{i-\frac{1}{2}} K_{-(i-\frac{1}{2})}^{-1})^{\pm 1}, \\ e_i &:= E_{i-\frac{1}{2}} + p^{-\delta_{i,1}} F_{-(i-\frac{1}{2})} K_{i-\frac{1}{2}}^{-1}, \\ f_i &:= E_{-(i-\frac{1}{2})} + p^{\delta_{i,1}} K_{-(i-\frac{1}{2})}^{-1} F_{i-\frac{1}{2}}, \quad i \in \mathbb{J}. \end{aligned}$$

When we want to emphasize the integer r , we denote this subalgebra by \mathbf{U}_r^j instead of \mathbf{U}^j .

The \mathbf{U}^j has the following defining relations: for $i, j \in \mathbb{I}$,

$$\begin{aligned}
 (2) \quad & k_i k_i^{-1} = k_i^{-1} k_i = 1, \\
 & k_i k_j = k_j k_i, \\
 & k_i e_j k_i^{-1} = q^{(\alpha_i - \alpha_{-i}, \alpha_j)} e_j, \\
 & k_i f_j k_i^{-1} = q^{-(\alpha_i - \alpha_{-i}, \alpha_j)} f_j, \\
 & e_i f_j - f_j e_i = \delta_{i,j} \frac{k_i - k_i^{-1}}{q - q^{-1}} \quad \text{if } (i, j) \neq (1, 1), \\
 & e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0 \quad \text{if } |i - j| = 1, \\
 & f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 = 0 \quad \text{if } |i - j| = 1, \\
 & e_i e_j - e_j e_i = 0 \quad \text{if } |i - j| > 1, \\
 & f_i f_j - f_j f_i = 0 \quad \text{if } |i - j| > 1, \\
 & e_1^2 f_1 - (q + q^{-1}) e_1 f_1 e_1 + f_1 e_1^2 = -(q + q^{-1}) e_1 (pqk_1 + p^{-1}q^{-1}k_1^{-1}), \\
 & f_1^2 e_1 - (q + q^{-1}) f_1 e_1 f_1 + e_1 f_1^2 = -(q + q^{-1}) (pqk_1 + p^{-1}q^{-1}k_1^{-1}) f_1.
 \end{aligned}$$

Also, \mathbf{U}^j is a right coideal of \mathbf{U} , that is, $\Delta(\mathbf{U}^j) \subset \mathbf{U}^j \otimes \mathbf{U}$. Indeed, we have

$$\begin{aligned}
 (3) \quad & \Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1}, \\
 & \Delta(e_i) = e_i \otimes K_{i-\frac{1}{2}}^{-1} + 1 \otimes E_{i-\frac{1}{2}} + p^{-\delta_{i,1}} k_i^{-1} \otimes F_{-(i-\frac{1}{2})} K_{i-\frac{1}{2}}^{-1}, \\
 & \Delta(f_i) = f_i \otimes K_{-(i-\frac{1}{2})}^{-1} + 1 \otimes E_{-(i-\frac{1}{2})} + p^{\delta_{i,1}} k_i \otimes K_{-(i-\frac{1}{2})}^{-1} F_{i-\frac{1}{2}} \quad \text{for } i \in \mathbb{I}^j.
 \end{aligned}$$

This fact enables us to regard the tensor product $M \otimes N$ of a \mathbf{U}^j -module M and a \mathbf{U} -module N as a \mathbf{U}^j -module. Thanks to the coassociativity of Δ , we have a natural isomorphism $M \otimes (N_1 \otimes N_2) \simeq (M \otimes N_1) \otimes N_2$ of \mathbf{U}^j -modules, where N_1 and N_2 are \mathbf{U} -modules.

Proposition 2.1.1. (1) *There exists a unique \mathbb{Q} -algebra automorphism ψ^j of \mathbf{U}^j which maps e_i, f_i, k_i, p, q to $e_i, f_i, k_i^{-1}, p^{-1}, q^{-1}$, respectively.*

(2) *There exists a unique $\mathbb{Q}(p, q)$ -algebra anti-automorphism σ^j of \mathbf{U}^j which maps e_i, f_i, k_i to f_i, e_i, k_i , respectively.*

Proof. These assertions are easily verified by the defining relations (2) of \mathbf{U}^j . \square

For notational simplicity, we write \bar{x} instead of $\psi^j(x)$ for $x \in \mathbf{U}^j$; it should be noted that ψ^j is different from the restriction of the bar-involution of \mathbf{U} , which we will not use in this paper.

2.2. Triangular decomposition of \mathbf{U}^j . Recall Lusztig's braid group actions on \mathbf{U} .

Definition 2.2.1 ([Lu94, Chapter 37]). Let $e \in \{1, -1\}$. For each $i \in \mathbb{I}$, define four automorphisms $T'_{i,e}$ and $T''_{i,-e}$ on \mathbf{U} by:

$$\begin{aligned}
 T'_{i,e}(E_j) &= \begin{cases} -K_i^e F_i & \text{if } j = i, \\ E_j & \text{if } |i - j| > 1, \\ [E_j, E_i]_e & \text{if } |i - j| = 1, \end{cases} & T'_{i,e}(F_j) &= \begin{cases} -E_i K_i^{-e} & \text{if } j = i, \\ F_j & \text{if } |i - j| > 1, \\ [F_i, F_j]_{-e} & \text{if } |i - j| = 1, \end{cases} \\
 T''_{i,-e}(E_j) &= \begin{cases} -F_i K_i^{-e} & \text{if } j = i, \\ E_j & \text{if } |i - j| > 1, \\ [E_i, E_j]_e & \text{if } |i - j| = 1, \end{cases} & T''_{i,-e}(F_j) &= \begin{cases} -K_i^e E_i & \text{if } j = i, \\ F_j & \text{if } |i - j| > 1, \\ [F_j, F_i]_{-e} & \text{if } |i - j| = 1, \end{cases} \\
 T'_{i,e}(K_j) = T''_{i,-e}(K_j) &= \begin{cases} K_i^{-1} & \text{if } j = i, \\ K_j & \text{if } |i - j| > 1, \\ K_i K_j & \text{if } |i - j| = 1; \end{cases}
 \end{aligned}$$

here we set $[X, Y]_e := XY - q^e YX$. By [Lu94, Theorem 39.4.3], for each $e \in \{1, -1\}$, the families $\{T'_{i,e} \mid i \in \mathbb{I}\}$ and $\{T''_{i,-e} \mid i \in \mathbb{I}\}$ both satisfy the braid relation of type A_{2r} . Let $W(\mathbb{I})$ denote the Weyl group of type A_{2r} with simple reflections $\{s_i \mid i \in \mathbb{I}\}$. Then the PBW-type basis of \mathbf{U} is described as follows.

Definition 2.2.2. Let $\mathbf{i} = (i_1, \dots, i_N)$ be a reduced word for the longest element $w_0 \in W(\mathbb{I})$. The root vectors $F_j(\mathbf{i})$, $j = 1, \dots, N$, associated with \mathbf{i} are given by:

$$F_1(\mathbf{i}) = F_{i_1}, \quad F_j(\mathbf{i}) = T''_{i_1,1} \cdots T''_{i_{j-1},1}(F_{i_j}).$$

For each positive root α , we set $F_\alpha(\mathbf{i}) := F_j(\mathbf{i})$ if $\alpha = s_{i_1} \cdots s_{i_{j-1}}(\alpha_j)$.

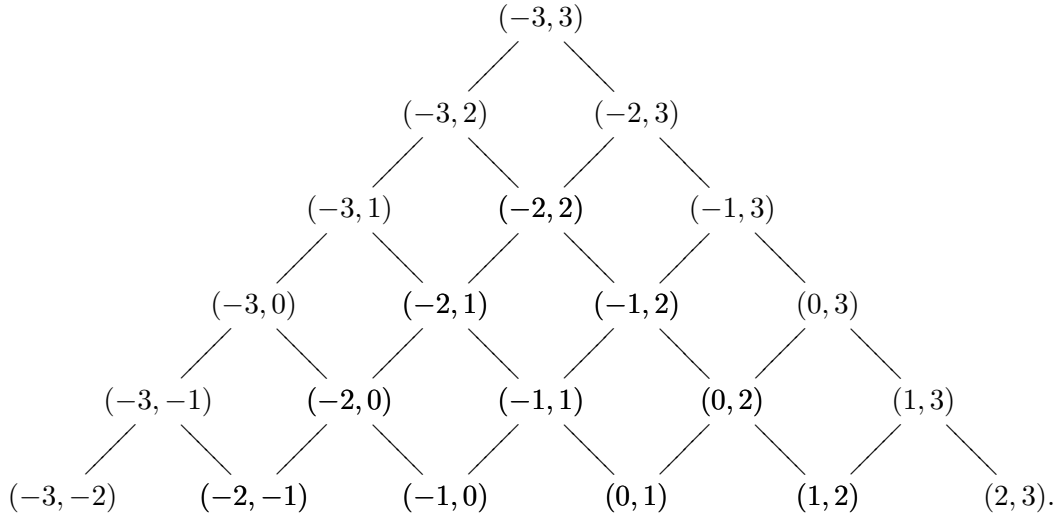
Theorem 2.2.3 ([Lu90, 4.2]). *Let $\mathbf{i} = (i_1, \dots, i_N)$ be a reduced word for $w_0 \in W(\mathbb{I})$. Then, the ordered monomials in the root vectors associated with \mathbf{i} form a linear basis of \mathbf{U}^- .*

We define a filtration of \mathbf{U}^j by setting $\deg(e_i) = \deg(f_i) = 1$ and $\deg(k_i) = 0$ for $i \in \mathbb{I}^j$. Then, by the defining relation (equation (2)) of \mathbf{U}^j , the associated graded algebra $\text{gr } \mathbf{U}^j$ is isomorphic to $\mathbf{U}^- \otimes \mathbf{U}^{j,0}$, where $\mathbf{U}^{j,0}$ denotes the subalgebra of \mathbf{U}^j generated by $k_i^{\pm 1}$, $i \in \mathbb{I}^j$. We define a linear isomorphism $\text{gr} : \mathbf{U}^j \rightarrow \mathbf{U}^- \otimes \mathbf{U}^{j,0}$ to be the composite map of the linear isomorphism $\mathbf{U}^j \simeq \text{gr } \mathbf{U}^j$ and the algebra isomorphism $\text{gr } \mathbf{U}^j \simeq \mathbf{U}^- \otimes \mathbf{U}^{j,0}$.

Recall that $\Phi_+ = \{\epsilon_i - \epsilon_j \mid -r \leq i < j \leq r\}$ denotes the set of positive roots of Φ with respect to the simple roots $\Pi = \{\epsilon_i - \epsilon_{i+1} \mid -r \leq i < r\}$. We decompose Φ_+ into three parts as:

$$\begin{aligned} \Phi_+ &= \Phi_{<0} \sqcup \Phi_0 \sqcup \Phi_{>0}, \\ \Phi_{<0} &:= \{\epsilon_i - \epsilon_j \mid i + j < 0\}, \\ \Phi_0 &:= \{\epsilon_i - \epsilon_j \mid i + j = 0\}, \\ \Phi_{>0} &:= \{\epsilon_i - \epsilon_j \mid i + j > 0\}. \end{aligned}$$

For example, when $r = 3$, the positive roots are displayed as follows:



Here, (i, j) means $\epsilon_i - \epsilon_j$. Then, the roots in Φ_0 lie on the vertical line through $(-3, 3)$, those in $\Phi_{<0}$ on the left of the line, and those in $\Phi_{>0}$ on the right.

Here, we recall the notion of reflection orders (or convex orders).

Definition 2.2.4. A total order \preceq on Φ_+ is said to be a reflection order if it satisfies the following: for each $\alpha, \beta \in \Phi_+$ and $a, b \in \mathbb{R}_{>0}$, if $a\alpha + b\beta \in \Phi_+$ and $\alpha \prec \beta$, then $\alpha \prec a\alpha + b\beta \prec \beta$.

Proposition 2.2.5 ([D93, Proposition 2.13]). *Let $\mathbf{i} = (i_1, \dots, i_N)$ be a reduced word for $w_0 \in W(\mathbb{I})$. Set $\alpha_j(\mathbf{i}) := s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$. Then, the total order \preceq on Φ_+ defined by $\alpha_1(\mathbf{i}) \prec \cdots \prec \alpha_N(\mathbf{i})$ is a reflection order. Moreover, this correspondence gives a bijection between the set of reduced words for $w_0 \in W(\mathbb{I})$ and the set of reflection orders on Φ^+ .*

Lemma 2.2.6. *There exists a reflection order \preceq on Φ_+ such that*

$$(4) \quad \Phi_{<0} \prec \Phi_0 \prec \Phi_{>0}.$$

Here, for subsets $A, B \subset \Phi_+$, $A \prec B$ means that $\alpha \prec \beta$ for all $\alpha \in A$ and $\beta \in B$.

Proof. See the next example. \square

Example 2.2.7. For simplicity, we write (i, j) instead of $\epsilon_i - \epsilon_j$ for $i < j$. We decompose $\Phi_{<0}$ into $\Phi_{<0,-} := \{(i, j) \in \Phi_{<0} \mid j \leq 0\}$ and $\Phi_{<0,+} := \{(i, j) \in \Phi_{<0} \mid j > 0\}$. Similarly, we set $\Phi_{>0,-} := \{(i, j) \in \Phi_{>0} \mid i < 0\}$ and $\Phi_{>0,+} := \{(i, j) \in \Phi_{>0} \mid i \geq 0\}$. Let us define a total order \preceq on Φ_+ by:

- (1) $\Phi_{<0,-} \prec \Phi_{<0,+} \prec \Phi_0 \prec \Phi_{>0,-} \prec \Phi_{>0,+}$;
- (2) for $(i, j), (i', j') \in \Phi_{<0,-}$, $(i, j) \prec (i', j')$ if and only if $i < i'$ or $(i = i' \text{ and } j < j')$;
- (3) for $(i, j), (i', j') \in \Phi_{<0,+}$, $(i, j) \prec (i', j')$ if and only if $j < j'$ or $(j = j' \text{ and } i < i')$;
- (4) for $(i, j), (i', j') \in \Phi_0$, $(i, j) \prec (i', j')$ if and only if $j < j'$;
- (5) for $(i, j), (i', j') \in \Phi_{>0,-}$, $(i, j) \prec (i', j')$ if and only if $i < i'$ or $(i = i' \text{ and } j < j')$;
- (6) for $(i, j), (i', j') \in \Phi_{>0,+}$, $(i, j) \prec (i', j')$ if and only if $j < j'$ or $(j = j' \text{ and } i < i')$.

The \preceq is a reflection order on Φ_+ satisfying $\Phi_{<0} \prec \Phi_0 \prec \Phi_{>0}$; the proof is straightforward.

For example, when $r = 3$, this total order is given as follows:

$$\begin{aligned} &(-3, -2) \prec (-3, -1) \prec (-3, 0) \prec (-2, -1) \prec (-2, 0) \prec (-1, 0) \\ &\prec (-3, 1) \prec (-2, 1) \prec (-3, 2) \\ &\prec (-1, 1) \prec (-2, 2) \prec (-3, 3) \\ &\prec (-2, 3) \prec (-1, 2) \prec (-1, 3) \\ &\prec (0, 1) \prec (0, 2) \prec (1, 2) \prec (0, 3) \prec (1, 3) \prec (2, 3). \end{aligned}$$

Fix a reflection order \preceq satisfying condition (4) in Lemma 2.2.6. Let \mathbf{i} be the reduced word for $w_0 \in W(\mathbb{I})$ corresponding to \preceq under the bijection of Proposition 2.2.5. We set $F_{i,j} := F_{\epsilon_i - \epsilon_j}(\mathbf{i})$ for $-r \leq i < j \leq r$. For each i, j , define $F'_{i,j} := \text{gr}^{-1}(F_{i,j})$, and set

$$f_{-j,-i} := F'_{i,j} \text{ if } i + j < 0, \quad h_i := F'_{-i,i}, \quad e_{i,j} := F'_{i,j} \text{ if } i + j > 0.$$

Let us compute some of these vectors. By [LS91, Lemma 1] (with a slight modification), we have

$$\begin{aligned} F_{i-1,j} &= [F_{i,j}, F_{i-\frac{1}{2}}]_1 \quad \text{if } (i-1, i) \prec (i, j), \\ F_{i,j+1} &= [F_{j+\frac{1}{2}}, F_{i,j}]_1 \quad \text{if } (i, j) \prec (j, j+1). \end{aligned}$$

In particular, by condition (4),

$$F_{-1,1} = [F_{\frac{1}{2}}, F_{-\frac{1}{2}}]_1, \quad F_{-(i+1),i+1} = \left[[F_{(i+\frac{1}{2})}, F_{-i,i}]_1, F_{-(i+\frac{1}{2})} \right]_1 \quad \text{for } 1 \leq i \leq r-1.$$

Applying gr^{-1} , we obtain

$$(5) \quad h'_1 = [e_1, f_1]_1, \quad h'_{i+1} = [[e_{i+1}, h_i]_1, f_{i+1}]_1.$$

This shows that the h_i 's are independent of the choice of a reflection order \preceq satisfying condition (4) in Lemma 2.2.6.

Let $\mathbf{U}_{<0}^j$ (resp., $\mathbf{U}_0^j, \mathbf{U}_{>0}^j$) denote the subspace of \mathbf{U}^j spanned by all ordered monomials in $f_{-j,-i}$ (resp., $h_i, e_{i,j}$). Then, we have an isomorphism of vector spaces

$$\mathbf{U}^j \simeq \mathbf{U}_{<0}^j \otimes \left(\mathbf{U}_0^j \otimes \mathbf{U}^{j,0} \right) \otimes \mathbf{U}_{>0}^j.$$

We call this linear isomorphism the triangular decomposition of \mathbf{U}^j associated with the reflection order \preceq , and $\mathbf{U}_{<0}^j$ (resp., $\mathbf{U}_0^j \otimes \mathbf{U}^{j,0}, \mathbf{U}_{>0}^j$) the negative part (resp., Cartan part, positive part) of \mathbf{U}^j . The triangular decomposition enables us to establish an analog of highest weight theory for the representation theory of \mathbf{U}^j .

Remark 2.2.8. Unlike the ordinary triangular decomposition of a quantized enveloping algebra, the negative part, the Cartan part, and the positive part of \mathbf{U}^J are just subspaces, not subalgebras. In addition, the negative part and the positive part may depend on the choice of a reflection order.

2.3. Verma modules and their irreducible quotients. Recall that $\mathbb{R}^{2r+1} = \bigoplus_{i=-r}^r \mathbb{R}\epsilon_i$ is the Euclidean space with standard basis $\{\epsilon_i \mid -r \leq i \leq r\}$ with respect to the inner product (\cdot, \cdot) , and $\alpha_i = \epsilon_{i-\frac{1}{2}} - \epsilon_{i+\frac{1}{2}}$, $i \in \mathbb{I}$, are the simple roots. Set $\beta_i := \alpha_{i-\frac{1}{2}} - \alpha_{-(i-\frac{1}{2})} = \epsilon_{i-1} - \epsilon_i - \epsilon_{-i} + \epsilon_{-(i-1)}$ for $i \in \mathbb{I}^J$.

Definition 2.3.1. Let $J \subset \mathbb{R}^{2r+1} := \{\lambda \in \mathbb{R}^{2r+1} \mid (\beta_i, \lambda) = 0 \text{ for all } i \in \mathbb{I}^J\}$. Then the bilinear form (\cdot, \cdot) on $\mathbb{R}^{2r+1} \times \mathbb{R}^{2r+1}$ induces a bilinear map $(\bigoplus_{i \in \mathbb{I}^J} \mathbb{R}\beta_i) \times (\mathbb{R}^{2r+1}/J) \rightarrow \mathbb{R}$, which we also denote by (\cdot, \cdot) . For each $i \in \mathbb{I}^J$, there exists a unique $\delta_i \in \mathbb{R}^{2r+1}/J$ such that

$$(\beta_j, \delta_i) = \delta_{i,j} \quad \text{for } i, j \in \mathbb{I}^J.$$

Set $\Lambda^J := \sum_{i \in \mathbb{I}^J} \mathbb{Z}\delta_i$ and $\Lambda_+^J := \sum_{i \in \mathbb{I}^J} \mathbb{Z}_{\geq 0}\delta_i$. Also, we set $\gamma_i := \epsilon_{i-1} - \epsilon_i + J \in \Lambda^J$.

By the definitions, we have

$$(\beta_i, \gamma_j) = (\alpha_i - \alpha_{-i}, \alpha_j) = \begin{cases} 3 & \text{if } i = j = 1, \\ 2 & \text{if } i = j \neq 1, \\ -1 & \text{if } |i - j| = 1, \\ 0 & \text{if } |i - j| > 1. \end{cases}$$

Define a partial order \leq on Λ^J by:

$$(6) \quad \mu \leq \lambda \text{ if and only if } \lambda - \mu \in \sum_{i \in \mathbb{I}^J} \mathbb{Z}_{\geq 0}\gamma_i.$$

For a \mathbf{U}^J -module M and $m \in M$, we say that m is of weight $\lambda \in \Lambda^J$ if it satisfies

$$k_i m = q^{(\beta_i, \lambda)} m$$

for all $i \in \mathbb{I}^J$; we denote by M_λ the subspace consisting of all $m \in M$ of weight λ .

Lemma 2.3.2. Let M be a \mathbf{U}^J -module and $\lambda \in \Lambda^J$. For each $i \in \mathbb{I}^J$, we have

$$f_i(M_\lambda) \subset M_{\lambda - \gamma_i}, \quad e_i(M_\lambda) \subset M_{\lambda + \gamma_i}.$$

Proof. This follows immediately from the relations $k_i f_j k_i^{-1} = q^{(\beta_i, -\gamma_j)} f_j$ and $k_i e_j k_i^{-1} = q^{(\beta_i, \gamma_j)} e_j$. \square

Recall the triangular decomposition of \mathbf{U}

$$\mathbf{U}^J \simeq \mathbf{U}_{<0}^J \otimes \left(\mathbf{U}_0^J \otimes \mathbf{U}^{J,0} \right) \otimes \mathbf{U}_{>0}^J,$$

and the root vectors $f_{-j,-i}, h_i, e_{i,j}$ associated with a reflection order satisfying condition (4) in Lemma 2.2.6.

Definition 2.3.3. Let $\lambda \in \Lambda^J$ and $H_i \in \mathbb{Q}(p, q)$, $i = 1, 2, \dots, r$. The Verma module $V(\lambda; \mathbf{H})$ over \mathbf{U}^J with highest weight λ associated with $\mathbf{H} := (H_1, \dots, H_r) \in \mathbb{Q}(p, q)^r$ is defined to be

$$V(\lambda; \mathbf{H}) := \mathbf{U}^J / I(\lambda; \mathbf{H}),$$

where $I(\lambda; \mathbf{H})$ denotes the left ideal of \mathbf{U}^J generated by $\mathbf{U}_{>0}^J$ and $k_i - q^{(\beta_i, \lambda)}$, $h_i - H_i$ for $i \in \mathbb{I}^J$.

By the triangular decomposition of \mathbf{U}^J , the Verma module $V(\lambda; \mathbf{H})$ has a unique maximal submodule, and hence, it has a unique irreducible quotient. We denote it by $L(\lambda; \mathbf{H})$ and call it the irreducible highest weight \mathbf{U}^J -module with highest weight λ associated with \mathbf{H} , or simply, with highest weight $(\lambda; \mathbf{H})$.

Definition 2.3.4. A nonzero \mathbf{U}^j -module M is called a highest weight module with highest weight $(\lambda; \mathbf{H}) \in \Lambda^j \times \mathbb{Q}(p, q)^r$ if there exists $m \in M_\lambda$ such that $\mathbf{U}_{>0}^j m = 0$, $h_i m = H_i m$ for $i \in \mathbb{I}^j$, and $M = \mathbf{U}^j m$. We call such an m a highest weight vector of M with highest weight $(\lambda; \mathbf{H})$.

Our definition of highest weight modules over \mathbf{U}^j depends on the choice of a reflection order satisfying condition (4) in Lemma 2.2.6. However, their \mathbf{U}^j -module structure is independent of such a choice, as we explain below.

Let M be a highest weight \mathbf{U}^j -module with highest weight $(\lambda; \mathbf{H})$ associated with a reflection order \preceq . Take another reflection order \preceq' , and denote the corresponding root vectors by $f'_{i,j}, h'_i, e'_{i,j}$. Then, we see from equation (5) that $h'_i = h_i$. Also, by the triangular decomposition associated with \prec , we have

$$e'_{i,j} \in \sum_{\substack{\nu, \mu \in \Lambda_+^j \\ \nu \prec \mu}} (\mathbf{U}_{<0}^j)_{-\nu} \otimes (\mathbf{U}_0^j \otimes \mathbf{U}^{j,0}) \otimes (\mathbf{U}_{>0}^j)_\mu;$$

here, $(\mathbf{U}_{<0}^j)_{-\nu} := \{x \in \mathbf{U}_{<0}^j \mid k_i x k_i^{-1} = q^{(\beta_i, -\nu)} x \text{ for all } i \in \mathbb{I}^j\}$, and define $(\mathbf{U}_{>0}^j)_\mu$ similarly. Therefore, it holds that $e'_{i,j} v = 0$ for all i, j . In addition, by expanding $f_{i,j}$ in ordered monomials in $f'_{i,j}, h_i, e'_{i,j}$, we see that $f_{i,j} v$ is a linear combination of $f'_{i,j} v$'s. From these, we conclude that M is a highest weight module with highest weight vector $(\lambda; \mathbf{H})$ associated with \preceq' . In particular, if we denote Verma modules and their irreducible quotients associated with \preceq' by $V'(\cdot; \cdot)$ and $L'(\cdot; \cdot)$, respectively, then we have

$$V(\lambda; \mathbf{H}) = V'(\lambda; \mathbf{H}), \quad L(\lambda; \mathbf{H}) = L'(\lambda; \mathbf{H}).$$

Hence, in this paper, we use only the reflection order given in Example 2.2.7.

Let $\mathcal{O}_{\text{int}}^j$ denote the category of all \mathbf{U}^j -modules M satisfying the following:

- (M1) M is decomposed into weight spaces, i.e., $M = \bigoplus_{\lambda \in \Lambda^j} M_\lambda$.
- (M2) Each weight space is finite-dimensional.
- (M3) There exist finitely many weights $\mu_1, \dots, \mu_n \in \Lambda^j$ such that each weight $\lambda \in \Lambda^j$ for which $M_\lambda \neq 0$ satisfies $\lambda \leq \mu_i$ for some $i = 1, \dots, n$.
- (M4) e_i and f_i act on M locally nilpotently, that is, for each $m \in M$, there exists $N \in \mathbb{N}$ such that $e_i^N m = 0 = f_i^N m$.

Note that Verma modules and their irreducible quotients are not necessarily objects of $\mathcal{O}_{\text{int}}^j$.

3. THE CASE $r = 1$

3.1. Classification of the irreducible modules in $\mathcal{O}_{\text{int}}^j$. We introduce some more notation.

Definition 3.1.1. (1) For $n \in \mathbb{Z}$, $[n] := \frac{q^n - q^{-n}}{q - q^{-1}}$.

(2) For $n \in \mathbb{Z}_{>0}$, $[n]! := \prod_{i=1}^n [i]$; we set $[0]! := 1$.

(3) For $x \in \mathbf{U}$ and $n \in \mathbb{Z}_{>0}$, $x^{(n)} := \frac{x^n}{[n]!}$; we set $x^{(0)} := 1$, and $x^{(n)} := 0$ if $n < 0$.

(4) For $x, y \in \mathbf{U}$ and $a \in \mathbb{Z}$, $[x, y]_a := xy - q^a yx$.

(5) For an invertible element h , $\{h\} := h + h^{-1}$.

(6) For an integer $n \in \mathbb{Z}$, $\{n\} := \{pq^n\} = pq^n + p^{-1}q^{-n}$.

In the case $r = 1$, the root vectors are

$$f_{0,1} = f_1, \quad h_1 = [e_1, f_1]_1, \quad e_{0,1} = e_1.$$

Lemma 3.1.2. In \mathbf{U}_1^j , we have

$$[h_1, f_1]_{-1} = -[2]\{pqk_1\}f_1, \quad [e_1, h_1]_{-1} = -[2]e_1\{pqk_1\}.$$

Proof. By equation (2). □

Lemma 3.1.3. *For each $n \in \mathbb{Z}_{\geq 0}$, we have*

$$e_1 f_1^{(n)} = f_1^{(n-1)} (h_1 - [n-1]\{pq^{-n}k_1\}) + q^n f_1^{(n)} e_1.$$

Proof. We prove the assertion by induction on n . This is trivial when $n = 0$. Assume that the assertion holds for a fixed $n \in \mathbb{Z}_{\geq 0}$. Then, we compute as follows:

$$\begin{aligned} e_1 f_1^{(n+1)} &= \frac{1}{[n+1]} e_1 f_1^{(n)} f_1 \\ &= \frac{1}{[n+1]} \left(f_1^{(n-1)} (h_1 - [n-1]\{pq^{-n}k_1\}) + q^n f_1^{(n)} e_1 \right) f_1 \\ &= \frac{1}{[n+1]} \left(f_1^{(n-1)} (q^{-1}f_1 h_1 - [2]\{pqk_1\}f_1 - [n-1]\{pq^{-n}k_1\}f_1) + q^n f_1^{(n)} (h_1 + qf_1 e_1) \right) \\ &= \frac{1}{[n+1]} \left(f_1^{(n-1)} (q^{-1}f_1 h_1 - [2]f_1\{pq^{-2}k_1\} - [n-1]f_1\{pq^{-n-3}k_1\}) + q^n f_1^{(n)} (h_1 + qf_1 e_1) \right) \\ &= \frac{1}{[n+1]} f_1^{(n)} (q^{-1}[n]h_1 - [2][n]\{pq^{-2}k_1\} - [n-1][n]\{pq^{-n-3}k_1\} + q^n h_1) + q^{n+1} f_1^{(n+1)} e_1 \\ &= f_1^{(n)} (h_1 - [n]\{pq^{-n-1}k_1\}) + q^{n+1} f_1^{(n+1)} e_1; \end{aligned}$$

the second equality follows from our inductive hypothesis, the third from Lemma 3.1.2, and the rest is straightforward. This proves the lemma. \square

Note that when $r = 1$, we have $\Lambda^J = \mathbb{Z}\delta_1$ and $\gamma_1 = 3\delta_1$. Let $M \in \mathcal{O}_{\text{int}}^J$. By the definition of $\mathcal{O}_{\text{int}}^J$, there exists $a \in \mathbb{Z}$ such that $M_{a\delta_1} \neq \{0\}$ and $M_{(a+3)\delta_1} = \{0\}$. Since the action of h_1 preserves weights, it defines a linear endomorphism of $M_{a\delta_1}$. In order to consider the Jordan canonical form for the action of h_1 on $M_{a\delta_1}$, we extend the base field $\mathbb{Q}(p, q)$ to its algebraic closure $\overline{\mathbb{Q}(p, q)}$ until the proof of Proposition 3.1.4. Let us write the Jordan canonical form as:

$$\begin{pmatrix} J_{d_1}(\mu_1) & & & \\ & J_{d_2}(\mu_2) & & \\ & & \ddots & \\ & & & J_{d_m}(\mu_m) \end{pmatrix},$$

where $J_{d_i}(\mu_i)$ denotes the Jordan block of size d_i whose eigenvalue is $\mu_i \in \overline{\mathbb{Q}(p, q)}$. We take a basis $\{v_{j,k} \mid j = 1, \dots, m, k = 1, \dots, d_j\}$ of $M_{a\delta_1}$ in such a way that

$$h_1 v_{j,k} = \mu_j v_{j,k} + v_{j,k-1}$$

for all $j = 1, \dots, m$, $k = 1, \dots, d_j$, where $v_{j,0} := 0$. By Lemma 3.1.3, we have

$$(7) \quad e_1 f_1^{(n)} v_{j,k} = (\mu_j - [n-1]\{a-n\}) f_1^{(n-1)} v_{j,k} + f_1^{(n-1)} v_{j,k-1}.$$

Proposition 3.1.4. *We have $\mu_j = [N_j]\{a - N_j - 1\}$ for some $N_j \in \mathbb{Z}_{\geq 0}$. In particular, each μ_j belongs to $\mathbb{Q}(p, q)$.*

Proof. Consider the case $k = 1$. By the local nilpotency of f_1 , there exists a unique nonnegative integer N_j such that

$$f_1^{(N_j)} v_{j,1} \neq 0 \text{ and } f_1^{(N_j+1)} v_{j,1} = 0.$$

Then, by equation (7), we have

$$0 = e_1 f_1^{(N_j+1)} v_{j,1} = (\mu_j - [N_j]\{a - N_j - 1\}) f_1^{(N_j)} v_{j,1}.$$

Since $f_1^{(N_j)} v_{j,1} \neq 0$, we conclude that $\mu_j = [N_j]\{a - N_j - 1\}$, as desired. \square

Proposition 3.1.5. *Each d_j is equal to 1, that is, h_1 is diagonalizable on $M_{a\delta_1}$.*

Proof. We use the notation N_j in the proof of Proposition 3.1.4. Assume, for a contradiction, that there exists $d_j > 1$. By equation (7), we have

$$e_1 f_1^{(n)} v_{j,2} = (\mu_j - [n-1]\{a-n\}) f_1^{(n-1)} v_{j,2} + f_1^{(n-1)} v_{j,1}$$

for all $n \geq 0$. Let N'_j denote the unique nonnegative integer such that

$$f_1^{(N'_j)} v_{j,2} \neq 0, \text{ and } f_1^{(N'_j+1)} v_{j,2} = 0.$$

When $N'_j > N_j$, we have

$$0 = (\mu_j - [N'_j]\{a - N'_j - 1\}) f_1^{(N'_j)} v_{j,2} + f_1^{(N'_j)} v_{j,1} = (\mu_j - [N'_j]\{a - N'_j - 1\}) f_1^{(N'_j)} v_{j,2}.$$

This implies that $\mu_j = [N'_j]\{a - N'_j - 1\} \neq [N_j]\{a - N_j - 1\}$, which causes a contradiction. When $N'_j = N_j$, we have

$$0 = (\mu_j - [N_j]\{a - N_j - 1\}) f_1^{(N_j)} v_{j,2} + f_1^{(N_j)} v_{j,1} = f_1^{(N_j)} v_{j,1}.$$

This contradicts the definition of N_j . When $N'_j < N_j$, we have

$$0 = (\mu_j - [N'_j]\{a - N'_j - 1\}) f_1^{(N'_j)} v_{j,2} + f_1^{(N'_j)} v_{j,1}.$$

Applying $e_i^{N'_j}$ on both sides, we obtain

$$0 = \prod_{l=1}^{N'_j+1} (\mu_j - [l-1]\{a-l\}) v_{j,2} + X v_{j,1} \quad \text{for some } X \in \mathbb{Q}(p, q).$$

Since the coefficient of $v_{j,2}$ is nonzero, this contradicts the linear independence of $v_{j,1}$ and $v_{j,2}$. This proves the proposition. \square

Theorem 3.1.6. *For each $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq 0}$, there exists a unique $(b+1)$ -dimensional irreducible \mathbf{U}_1^j -module $L(a; b) \in \mathcal{O}_{\text{int}}^j$ such that*

$$L(a; b) = \bigoplus_{n=0}^b v_n,$$

$$v_n = f_1^{(n)} v_0, \quad k_1 v_0 = q^a v_0, \quad h_1 v_0 = [b]\{a - b - 1\} v_0.$$

Conversely, each irreducible \mathbf{U}_1^j -module in $\mathcal{O}_{\text{int}}^j$ is isomorphic to $L(a; b)$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq 0}$.

Proof. It is straightforward to show that $L(a; b)$ is a $(b+1)$ -dimensional irreducible \mathbf{U}_1^j -module, and so we omit the details. Let $V \in \mathcal{O}_{\text{int}}^j$ be an irreducible \mathbf{U}_1^j -module. By the definition of $\mathcal{O}_{\text{int}}^j$, there exists an integer $a \in \mathbb{Z}$ such that $V_{a\delta_1} \neq 0$ and $e_1 V_{a\delta_1} = 0$. Also, by Propositions 3.1.4 and 3.1.5, there exist $b \in \mathbb{Z}_{\geq 0}$ and $v \in V_{a\delta_1} \setminus \{0\}$ such that $f_1^{(b)} v \neq 0$, $f_1^{(b+1)} v = 0$, and $h_1 v = [b]\{a - b - 1\} v$. Hence the \mathbf{U}_1^j -submodule generated by v is identical to $\bigoplus_{n=0}^b f_1^{(n)} v$, which is isomorphic to $L(a; b)$ by the definitions of v, a , and b . Since V is irreducible, we have $V = \mathbf{U}_1^j v \simeq L(a; b)$. This proves the theorem. \square

Note that $L(a; b)$ is the irreducible quotient $L(\lambda; \mathbf{H})$ of the Verma module $V(\lambda; \mathbf{H})$ with highest weight $(\lambda; \mathbf{H}) = (a\delta_1; [b]\{a - b - 1\})$. Hence, Theorem 3.1.6 gives a necessary and sufficient condition for $L(\lambda; \mathbf{H})$ to be an object of $\mathcal{O}_{\text{int}}^j$.

Corollary 3.1.7. *Let $a \in \mathbb{Z}$ and $H_1 \in \mathbb{Q}(p, q)$. Then, the irreducible highest weight module $L(a\delta_1; H_1)$ belongs to $\mathcal{O}_{\text{int}}^j$ if and only if $H_1 = [b]\{a - b - 1\}$ for some $b \in \mathbb{Z}_{\geq 0}$. Moreover, the assignment $(a, b) \mapsto [L(a; b)]$, where $[L(a; b)]$ denotes the isomorphism class of $L(a; b)$, gives a bijection from $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ to the set of isomorphism classes of irreducible \mathbf{U}_1^j -modules in $\mathcal{O}_{\text{int}}^j$.*

3.2. Complete reducibility. Set $z_1 := h_1 + \frac{[2]pq}{1-q^2}k_1 + \frac{[2]p^{-1}q^{-1}}{1-q^{-4}}k_1^{-1} \in \mathbf{U}_1^j$.

Lemma 3.2.1. *In \mathbf{U}_1^j , we have*

$$z_1 f_1 = q^{-1} f_1 z_1, \quad z_1 e_1 = q e_1 z_1.$$

Proof. By Lemma 3.1.2 and the equalities

$$[k_1, f_1]_{-1} = (1 - q^2)k_1 f_1, \quad [k_1^{-1}, f_1]_{-1} = (1 - q^{-4})k_1^{-1} f_1,$$

it follows that $z_1 f_1 = q^{-1} f_1 z_1$. Noting that z_1 is invariant under the anti-automorphism σ^j defined in Proposition 2.1.1 (2), we obtain the other equality. \square

Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq 0}$, and take a highest weight vector $v \in L(a; b)$. Then we have

$$z_1 f_1^{(n)} v = q^{-n} \left([b] \{a - b - 1\} + \frac{[2]pq^{1+a}}{1 - q^2} + \frac{[2]p^{-1}q^{-1-a}}{1 - q^{-4}} \right) v.$$

Denoting by $z_1(a, b, n)$ the coefficient of v on the right-hand side, one has

$$z_1(a, b, n) = -\frac{pq^{a-b-n}(q^{b+1} + q^{-b-1})}{q - q^{-1}} + \frac{p^{-1}q^{-a+2b-n+1}}{q - q^{-1}}.$$

Using this, one can verify that the function $\mathbb{Z}^3 \rightarrow \mathbb{Q}(p, q)$, $(a, b, n) \mapsto z_1(a, b, n)$, is injective.

Lemma 3.2.2. *Let $M \in \mathcal{O}_{\text{int}}^j$, $a, a' \in \mathbb{Z}$, and $b, b' \in \mathbb{Z}_{\geq 0}$. Then, each short exact sequence of the form*

$$(8) \quad 0 \rightarrow L(a; b) \xrightarrow{\iota} M \xrightarrow{\pi} L(a'; b') \rightarrow 0$$

splits.

Proof. Let $v \in L(a', b')$ be a highest weight vector, and take $u \in \pi^{-1}(v)$. Since \mathbf{U}_1^j -module homomorphisms preserve generalized eigenspaces of z_1 , we may assume that u is a generalized eigenvector of z_1 with eigenvalue $z_1(a', b', 0)$. Then, $e_1 u$ is a generalized eigenvector of z_1 with eigenvalue $z_1(a', b', -1)$. Since $\pi(e_1 u) = e_1 \pi(u) = e_1 v = 0$, it follows that $e_1 u \in \iota(L(a', b'))$. However, the eigenvalues of z_1 on $L(a, b)$ are $z_1(a, b, n)$, $0 \leq n \leq b$. Therefore, $e_1 u = 0$, and hence we obtain a section $v \mapsto u$ of π . This proves the lemma. \square

Now, the complete reducibility of \mathbf{U}^j -modules in $\mathcal{O}_{\text{int}}^j$ follows from a standard argument; see, for example, [HK02, Section 3.5].

Theorem 3.2.3. *Every \mathbf{U}_1^j -module in $\mathcal{O}_{\text{int}}^j$ is completely reducible.*

Corollary 3.2.4. *Let $M \in \mathcal{O}_{\text{int}}^j$. Then, M is decomposed into a direct sum of z_1 -eigenspaces with possible eigenvalues $z_1(a, b, n)$, $a \in \mathbb{Z}, 0 \leq n \leq b$. In particular, if $z_1 m = z_1(a, b, 0)$, then $e_1 m = 0$.*

4. COMPLETE REDUCIBILITY AND THE IRREDUCIBLE MODULES

Throughout this section, we fix $e \in \{1, -1\}$.

4.1. Braid group action on \mathbf{U}^j .

Definition 4.1.1. For $i \in \mathbb{I}^j \setminus \{1\}$, define two automorphisms $\tau'_{i,e}$ and $\tau''_{i,-e}$ of \mathbf{U}^j_r by:

$$\begin{aligned} \tau'_{i,e}(e_j) &= \begin{cases} -k_i^e f_i & \text{if } j = i, \\ e_j & \text{if } |i - j| > 1, \\ [e_j, e_i]_e & \text{if } |i - j| = 1, \end{cases} & \tau'_{i,e}(f_j) &= \begin{cases} -e_i k_i^{-e} & \text{if } j = i, \\ f_j & \text{if } |i - j| > 1, \\ [f_i, f_j]_{-e} & \text{if } |i - j| = 1, \end{cases} \\ \tau''_{i,-e}(e_j) &= \begin{cases} -f_i k_i^{-e} & \text{if } j = i, \\ e_j & \text{if } |i - j| > 1, \\ [e_i, e_j]_e & \text{if } |i - j| = 1, \end{cases} & \tau''_{i,-e}(f_j) &= \begin{cases} -k_i^e e_i & \text{if } j = i, \\ f_j & \text{if } |i - j| > 1, \\ [f_j, f_i]_{-e} & \text{if } |i - j| = 1, \end{cases} \\ \tau'_{i,e}(k_j) = \tau''_{i,-e}(k_j) &= \begin{cases} k_i^{-1} & \text{if } j = i, \\ k_j & \text{if } |i - j| > 1, \\ k_i k_j & \text{if } |i - j| = 1. \end{cases} \end{aligned}$$

Proposition 4.1.2. The $\tau'_{i,e}$ (resp., $\tau''_{i,-e}$), $i \in \mathbb{I}^j \setminus \{1\}$, are indeed automorphisms of \mathbf{U}^j . Moreover, they satisfy the braid relation of type A_{r-1} .

Proof. Set $\tau_i := \tau'_{i,e}$ (resp., $\tau''_{i,e}$), $i \in \mathbb{I}^j \setminus \{1\}$. We need to verify that the relations in (2) hold if we replace e_i, f_i, k_i by $\tau_j(e_i), \tau_j(f_i), \tau_j(k_i)$, respectively. By comparing Definition 4.1.1 with Definition 2.2.1, one immediately finds that the nontrivial assertions are

$$\begin{aligned} \tau_2(e_1)^2 \tau_2(f_1) - (q + q^{-1}) \tau_2(e_1) \tau_2(f_1) \tau_2(e_1) + \tau_2(f_1) \tau_2(e_1)^2 \\ = -(q + q^{-1}) \tau_2(e_1) (pq \tau_2(k_1) + p^{-1} q^{-1} \tau_2(k_1)^{-1}), \\ \tau_2(f_1)^2 \tau_2(e_1) - (q + q^{-1}) \tau_2(f_1) \tau_2(e_1) \tau_2(f_1) + \tau_2(e_1) \tau_2(f_1)^2 \\ = -(q + q^{-1}) (pq \tau_2(k_1) + p^{-1} q^{-1} \tau_2(k_1)^{-1}) \tau_2(f_1). \end{aligned}$$

These are checked by direct calculation, or by means of a computer program GAP [GAP16] with a package Quagroup (see [KP11, 4.5]). Also, one can verify the braid relation in the same way as for the braid group action on \mathbf{U} . This proves the proposition. \square

4.2. Braid group action on \mathbf{U}^j -modules. In this subsection, we define a braid group action on \mathbf{U}^j -modules in $\mathcal{O}^j_{\text{int}}$. Since the proofs of the propositions in this subsection are almost the same as those in the ordinary quantum group theory, we omit the details.

Definition 4.2.1. Let $M \in \mathcal{O}^j_{\text{int}}$. For each $i \in \mathbb{I}^j \setminus \{1\}$, we define two automorphisms $\tau'_{i,e}$ and $\tau''_{i,e}$ on M by:

$$\begin{aligned} \tau'_{i,e}(m) &= \sum_{\substack{a,b,c \in \mathbb{Z}_{\geq 0} \\ a-b+c=n}} (-q)^b q^{e(-ac+b)} f_i^{(a)} e_i^{(b)} f_i^{(c)} m, \\ \tau''_{i,e}(m) &= \sum_{\substack{a,b,c \in \mathbb{Z}_{\geq 0} \\ -a+b-c=n}} (-q)^b q^{e(-ac+b)} e_i^{(a)} f_i^{(b)} e_i^{(c)} m, \end{aligned}$$

where $n \in \mathbb{Z}$, and $m \in M$ is such that $k_i m = q^n m$.

Proposition 4.2.2 (see [Lu94, Proposition 5.2.2]). Let $M \in \mathcal{O}^j_{\text{int}}$, $i \in \mathbb{I}^j$, and let $\lambda \in \Lambda^j$ be such that $(\beta_i, \lambda) \geq 0$, $j \in \{0, 1, \dots, (\beta_i, \lambda)\}$; we set $h := (\beta_i, \lambda) - j$.

- (1) If $\eta \in M_\lambda$ is such that $e_i \eta = 0$, then $\tau'_{i,e}(f_i^{(j)} \eta) = (-1)^j q^{e(jh+j)} f_i^{(h)} \eta$.
- (2) If $\xi \in M_{-\lambda}$ is such that $f_i \xi = 0$, then $\tau''_{i,e}(e_i^{(j)} \xi) = (-1)^j q^{e(jh+j)} e_i^{(h)} \xi$.

Proposition 4.2.3 (see [Lu94, Proposition 5.2.3]). Let $M \in \mathcal{O}^j_{\text{int}}$, $i \in \mathbb{I}^j$, and $m \in M_\lambda$.

- (1) We have $\tau'_{i,e} \tau''_{i,-e} = \text{id}_M = \tau''_{i,-e} \tau'_{i,e}$.
- (2) We have $\tau''_{i,e}(m) = (-1)^{(\beta_i, \lambda)} q^{e(\beta_i, \lambda)} \tau'_{i,e}(m)$.

Proposition 4.2.4 (see [Lu94, Proposition 37.1.2]). *Let $M \in \mathcal{O}_{\text{int}}^j$ and $i \in \mathbb{I} \setminus \{1\}$. Then, for each $m \in M$ and $x \in \mathbf{U}_r^j$, we have*

$$\tau'_{i,e}(xm) = \tau'_{i,e}(x)\tau'_{i,e}(m), \quad \tau''_{i,e}(xm) = \tau''_{i,e}(x)\tau''_{i,e}(m).$$

In what follows, we write $\tau_i = \tau''_{i,1}$ for $i \in \mathbb{I} \setminus \{1\}$.

4.3. Classification of the irreducible modules in $\mathcal{O}_{\text{int}}^j$. Recall the triangular decomposition $\mathbf{U}^j = \mathbf{U}_{<0}^j \otimes (\mathbf{U}_0^j \otimes \mathbf{U}^{j,0}) \otimes \mathbf{U}_{>0}^j$ associated with the reflection order \preceq defined in Example 2.2.7. Also, recall from (5) in Section 2.3, the explicit form of the root vectors $h_i = \text{gr}^{-1}(F_{-i,i}) \in \mathbf{U}_0^j$, $i \in \mathbb{I} = \{1, \dots, r\}$. We remark that an irreducible highest weight module is determined by the eigenvalues of k_i 's and h_i 's for a highest weight vector. However, h_i 's are sometimes difficult to deal with.

Proposition 4.3.1. *Let $V(\lambda; \mathbf{H})$ be the Verma module with highest weight $(\lambda; \mathbf{H})$. Then, \mathbf{H} is determined by the $\tau_i \cdots \tau_2(h_1)$ -eigenvalue of v for $i \in \mathbb{I}$.*

Proof. For each $i \in \mathbb{I}$, set $\text{ef}(i) := e_i \cdots e_2 e_1 f_1 f_2 \cdots f_i$. By equation (5), the h_i is of the form

$$h_i = \sum_{\sigma \in \mathfrak{S}_{2i}} a_i(\sigma) x_{\sigma(1)} \cdots x_{\sigma(2i)},$$

where \mathfrak{S}_{2i} denotes the $2i$ -th symmetric group, $a_i(\sigma) \in \mathbb{Q}(q)$, $x_j = e_{i+1-j}$ for $1 \leq j \leq i$, and $x_j = f_{j-i}$ for $i+1 \leq j \leq 2i$. From this, noting that v is a highest weight vector, we deduce that $h_i v$ is of the form

$$h_i v = \left(\text{ef}(i) + \sum_{1 \leq i_1 \leq \dots \leq i_l < i} f_{i_1, \dots, i_l}(k_1, \dots, k_i) \text{ef}(i_1) \cdots \text{ef}(i_l) \right) v,$$

where $f_{i_1, \dots, i_l}(k_1, \dots, k_i) \in \mathbb{Q}(q)[k_1^{\pm 1}, \dots, k_i^{\pm 1}]$. Therefore, the h_i -eigenvalue H_i of v is determined by the $\text{ef}(j)$ -eigenvalue of v for $j \leq i$.

Also, $\tau_i \cdots \tau_2(h_1)$ is of the form

$$\tau_i \cdots \tau_2(h_1) = \sum_{\sigma \in \mathfrak{S}_{2i}} b_i(\sigma) x_{\sigma(1)} \cdots x_{\sigma(2i)},$$

where $b_i(\sigma) \in \mathbb{Q}(q)$. In the same way as above, the $\tau_i \cdots \tau_2(h_1)$ -eigenvalue of v is determined by the $\text{ef}(j)$ -eigenvalue of v for $j \leq i$. Conversely, the $\tau_j \cdots \tau_2(h_1)$ -eigenvalue of v for $j \leq i$ altogether determine the $\text{ef}(j)$ -eigenvalue of v for $j \leq i$, which, in turn, determine the h_i -eigenvalue H_i of v . This proves the proposition. \square

This proposition enables us to replace h_i with $\tau_i \cdots \tau_2(h_1)$ for $i \in \mathbb{I}$. Hence, from now on, we redefine h_i , $i \in \mathbb{I}$, as $h_1 = [e_1, f_1]_1$ and $h_i = \tau_i \cdots \tau_2(h_1)$.

Let $L \in \mathcal{O}_{\text{int}}^j$ be an irreducible \mathbf{U}^j -module. By condition (M3), there exists $\lambda \in \Lambda^j$ such that $L_\lambda \neq 0$ and $L_\mu = 0$ for all $\mu > \lambda$. By the case $r = 1$, h_1 acts on L_λ semisimply.

Lemma 4.3.2. *We have*

$$[h_1, h_2]_0 = [h_1, (q - q^{-1})(f_2[e_2, h_1]_1 - p^{-1}q^2 f_2 e_2 k_1^{-1})]_0 \in \mathbf{U}^j(e_2, e_2 h_1, e_2 h_1^2),$$

where $\mathbf{U}^j(e_2, e_2 h_1, e_2 h_1^2)$ denotes the left ideal of \mathbf{U}^j generated by $e_2, e_2 h_1, e_2 h_1^2$.

Proof. By direct calculation (or by using GAP). \square

This lemma implies that $[h_1, h_2]_0 L_\lambda = 0$; namely, the actions of h_1 and h_2 commute with each other on L_λ .

Lemma 4.3.3. *Let $i, j \in \mathbb{I}$. If $j \neq i, i+1$, then we have $\tau_j(h_i) = h_i$.*

Proof. The assertion in the case $j > i + 1$ follows from the definitions of τ_j and h_i . When $j < i$, by the braid relation for the τ_j 's, we see that

$$\begin{aligned}\tau_j(h_i) &= \tau_j(\tau_i \tau_{i-1} \cdots \tau_2)(h_1) \\ &= \tau_i \cdots \tau_{j+2} \tau_j \tau_{j+1} \tau_j \cdots \tau_2(h_1) \\ &= \tau_i \cdots \tau_{j+2} \tau_{j+1} \tau_j \tau_{j+1} \tau_{j-1} \cdots \tau_2(h_1) \\ &= \tau_i \cdots \tau_{j+2} \tau_{j+1} \tau_j \cdots \tau_2 \tau_{j+1}(h_1) \\ &= \tau_i \cdots \tau_2(h_1) = h_i.\end{aligned}$$

This proves the lemma. \square

Proposition 4.3.4. *Let $L \in \mathcal{O}_{\text{int}}^j$ be an irreducible module. Take $\lambda \in \Lambda^j$ such that $L_\lambda \neq 0$ and $L_\mu = 0$ for all $\mu > \lambda$. Then, the actions of h_1, \dots, h_r commute with each other on L_λ .*

Proof. Let $i, j \in \mathbb{I}^j$ be such that $j < i$. By Lemma 4.3.3,

$$[h_j, h_i]_0 = \tau_j \cdots \tau_2([h_1, h_i]_0) = \tau_j \cdots \tau_2 \tau_i \cdots \tau_3([h_1, h_2]_0).$$

Also, by Lemma 4.3.2,

$$\tau_j \cdots \tau_2 \tau_i \cdots \tau_3([h_1, h_2]_0) \in \mathbf{U}^j(\tau_{j,i}(e_2), \tau_{j,i}(e_2)h_j, \tau_{j,i}(e_2)h_j^2),$$

where $\tau_{j,i}$ denotes $\tau_j \cdots \tau_2 \tau_i \cdots \tau_3$. Since $\tau_{j,i}(e_2) \in \mathbf{U}_{>0}^j$, the vectors $\tau_{j,i}(e_2)h_j^l$, $l = 0, 1, 2$, act on L_λ by 0. This proves the proposition. \square

As a corollary of this proposition, we can take a simultaneous eigenvector $v \in L_\lambda$ for h_1, \dots, h_r . Let $H_i \in \mathbb{Q}(p, q)$ denote the eigenvalue of h_i . Then the submodule generated by v is a highest weight module with highest weight $(\lambda; H_1, \dots, H_r)$. Since L is irreducible, we conclude that L is a highest weight module.

Theorem 4.3.5. *Each irreducible module in $\mathcal{O}_{\text{int}}^j$ is a highest weight module with highest weight $(\lambda; \mathbf{H})$ for some $\lambda \in \Lambda$ and $\mathbf{H} = (H_1, \dots, H_r) \in \mathbb{Q}(p, q)^r$ satisfying the following:*

- (1) $a_i := (\beta_i, \lambda) \geq 0$ for each $i \in \mathbb{I}^j \setminus \{1\}$.
- (2) For each $i \in \mathbb{I}^j$, there exists $b_i \in \mathbb{Z}_{\geq 0}$ such that $0 \leq b_i \leq a_i$ and $H_i = [b_1 + \cdots + b_i]\{a_1 + \cdots + a_i - (b_1 + \cdots + b_i) - 1\}$; here, recall that $\{n\} = pq^n + p^{-1}q^{-n} \in \mathbb{Q}(p, q)$ for $n \in \mathbb{Z}$.

Proof. We have shown that each irreducible module in $\mathcal{O}_{\text{int}}^j$ is a highest weight module with highest weight $(\lambda; \mathbf{H})$ for some $\lambda \in \Lambda^j$ and $\mathbf{H} \in \mathbb{Q}(p, q)^r$. It is easy to verify that the irreducible highest weight module $L(\lambda; \mathbf{H})$ belongs to $\mathcal{O}_{\text{int}}^j$ if and only if $f_i^N v = 0$, $i \in \mathbb{I}^j$, for a sufficiently large N , where $v \in L(\lambda; \mathbf{H})$ is a highest weight vector. By the case $r = 1$, the equality $f_1^N v = 0$ is equivalent to the existence of $b_1 \in \mathbb{Z}_{\geq 0}$ satisfying the equality $H_1 = [b_1]\{a_1 - b_1 - 1\}$. Also, by the representation theory of $U_q(\mathfrak{sl}_2)$, the condition $f_i^N v = 0$, $i \geq 2$, is equivalent to $a_i \geq 0$.

It remains to determine the possible values of H_2, \dots, H_r . Let us assume the following:

- (*) Let $M \in \mathcal{O}_{\text{int}}^j$ be a \mathbf{U}_2^j -module, $v \in M$ a highest weight vector with highest weight $(a_1\delta_1 + a_2\delta_2; H_1, H_2)$. If $H_1 = [b_1]\{a_1 - b_1 - 1\}$ for some $b_1 \in \mathbb{Z}_{\geq 0}$, then $H_2 = [b_1 + b_2]\{a_1 + a_2 - (b_1 + b_2) - 1\}$ for some $0 \leq b_2 \leq a_2$.

In Section 6.1, we will prove that this assumption always holds (without assuming this theorem). Let $i \geq 3$, and assume that for all $j < i$, $H_j = [b_1 + \cdots + b_j]\{a_1 + \cdots + a_j - (b_1 + \cdots + b_j) - 1\}$ for some $0 \leq b_j \leq a_j$. Set $T_i := (\tau_{i-1}\tau_i) \cdots (\tau_3\tau_4)(\tau_2\tau_3)$, and consider the subalgebra $T_i(\mathbf{U}_2^j) \subset \mathbf{U}^j$. We have $T_i(k_1) = k_1 \cdots k_{i-1}$, $T_i(k_2) = k_i$, $T_i(h_1) = h_{i-1}$, and $T_i(h_2) = h_i$. If we regard L as a \mathbf{U}_2^j -module via the algebra homomorphism $T_i : \mathbf{U}_2^j \rightarrow \mathbf{U}_r^j$, the v is a highest weight vector with highest weight $((a_1 + \cdots + a_{i-1})\delta_1 + a_i\delta_2; H_{i-1}, H_i)$. By assumption (*), H_i must be of the form $[b_1 + \cdots + b_{i-1} + b_i]\{a_1 + \cdots + a_{i-1} + a_i - (b_1 + \cdots + b_{i-1} + b_i) - 1\}$ for some $0 \leq b_i \leq a_i$. This proves the theorem. \square

From now on, we write $L(\mathbf{a}; \mathbf{b})$ instead of $L(\lambda; \mathbf{H})$, where $\mathbf{a} = (a_1, \dots, a_r)$ and $\mathbf{b} = (b_1, \dots, b_r)$ are such that $a_i = (\beta_i, \lambda)$, $H_i = [b_1 + \cdots + b_i]\{(a_1 + \cdots + a_i) - (b_1 + \cdots + b_i) - 1\}$.

Corollary 4.3.6. *Let $\lambda \in \Lambda^j$ and $\mathbf{H} \in \mathbb{Q}(p, q)^r$. Then, the irreducible highest weight module $L(\lambda; \mathbf{H})$ belongs to $\mathcal{O}_{\text{int}}^j$ if and only if $L(\lambda; \mathbf{H}) = L(\mathbf{a}; \mathbf{b})$ for some $(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}^r \times \mathbb{Z}_{\geq 0}^r$ such that $a_i \geq b_i$, $i \in \mathbb{I}^j \setminus \{1\}$. Moreover, the assignment $(\mathbf{a}, \mathbf{b}) \mapsto [L(\mathbf{a}; \mathbf{b})]$, where $[L(\mathbf{a}; \mathbf{b})]$ denotes the isomorphism class of $L(\mathbf{a}; \mathbf{b})$, gives a bijection from $\{(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}^r \times \mathbb{Z}_{\geq 0}^r \mid a_i \geq b_i, i \in \mathbb{I}^j \setminus \{1\}\}$ to the set of isomorphism classes of irreducible \mathbf{U}^j -modules in $\mathcal{O}_{\text{int}}^j$.*

4.4. Complete reducibility. In this subsection only, we set $A := \mathbf{U}^j$, and write B for \mathbf{U}^j with p replaced by $p^{-1}q$. Consider the anti-algebra homomorphism $S : A \rightarrow B$ over $\mathbb{Q}(q)$ defined by:

$$S(e_i) = -e_i k_i, \quad S(f_i) = -k_i^{-1} f_i, \quad S(k_i) = k_i^{-1}, \quad S(p) = p^{-1} q.$$

It is easily checked that S is an anti-algebra homomorphism. In addition, S has the inverse:

$$S^{-1}(e_i) = -k_i e_i, \quad S^{-1}(f_i) = -f_i k_i^{-1}, \quad S^{-1}(k_i) = k_i^{-1}, \quad S^{-1}(p) = p^{-1} q.$$

For an A -module M , define a B -module $S_*(M) := M^\vee$ by:

$$(x \cdot g)(m) = g(S^{-1}(x) \cdot g) \quad \text{for } x \in B, g \in S(M), m \in M,$$

where M^\vee denotes the restricted dual of M , i.e., $M^\vee = \bigoplus_{\lambda \in \Lambda^j} \text{Hom}_{\mathbb{Q}(p, q)}(M_\lambda, \mathbb{Q}(p, q))$. Similarly, we associate an A -module $S^*(N)$ with each B -module N .

Lemma 4.4.1. *Let $L \in \mathcal{O}_{\text{int}}^j$ be the irreducible highest weight A -module with highest weight $(\lambda; \mathbf{H})$. Then, $S_*(L)$ is the irreducible lowest weight A -module with lowest weight $(-\lambda; \mathbf{H})$.*

Proof. Let $v \in L$ be a highest weight vector, and let $g \in S_*(L)$ be such that $g(v) = 1$ and $g(u) = 0$ for all $u \in L_\mu$, $\mu < \lambda$. Then, we have

$$(k_i g)(v) = g(k_i^{-1} v) = q^{-(\beta_i, \lambda)} g(v),$$

$$(h_i g)(v) = g(S^{-1}(h_i) v).$$

Since $S^{-1}(h_i) v \in L_\lambda = \mathbb{Q}(p, q) v$, we have $S^{-1}(h_i) v = H'_i v$ for some $H'_i \in \mathbb{Q}(p, q)$, and hence $h_i g = H'_i g$. Therefore, Bg is a lowest weight module with lowest weight $(-\lambda; H'_1, \dots, H'_r)$.

Now, it remains to show that $S_*(L)$ is irreducible. Suppose that $N \subset S_*(L)$ is a submodule. Then $S^*(N)$ is a submodule of $S^*(S_*(L)) \simeq L$. Since L is irreducible, $S^*(N)$ is identical either to 0 or to L , and hence N is identical either to 0 or to $S_*(L)$. Thus, $S_*(L)$ is irreducible. This proves the lemma. \square

Lemma 4.4.2. *Let M be an A -module. Suppose that M contains an irreducible submodule $L \simeq L(\lambda; \mathbf{H})$ for some $\lambda \in \Lambda^j$ and $\mathbf{H} \in \mathbb{Q}(p, q)^r$. Then, $M \simeq L \oplus (M/L)$.*

Proof. It suffices to show that the short exact sequence

$$0 \rightarrow L \xrightarrow{\iota} M \xrightarrow{\pi} M/L \rightarrow 0$$

splits. By the previous lemma, $S_*(M)$ has an irreducible submodule $S_*(L)$. Applying S^* to the inclusion $S_*(L) \hookrightarrow S_*(M)$, we obtain a surjection $M \twoheadrightarrow L$ of A -modules. Since the composite map $L \xrightarrow{\iota} M \twoheadrightarrow L$ is nonzero, it follows from Schur's lemma that this composite map is an isomorphism of A -modules. By composing the inverse of this isomorphism with the surjection $M \twoheadrightarrow L$, we obtain a retraction of ι . This proves the lemma. \square

Now, the complete reducibility of the \mathbf{U}^j -modules in $\mathcal{O}_{\text{int}}^j$ follows from a standard argument; see, for example, [HK02, Section 3.5].

Theorem 4.4.3. *Every \mathbf{U}^j -module in $\mathcal{O}_{\text{int}}^j$ is completely reducible.*

Corollary 4.4.4. *Every highest weight module in $\mathcal{O}_{\text{int}}^j$ is irreducible.*

Proof. Let M be a highest weight module in $\mathcal{O}_{\text{int}}^j$, and $v \in M$ a highest weight vector. By Theorem 4.4.3, we can decompose $M = \mathbf{U}^j v$ into the direct sum of irreducible submodules. Since the weight space of $\mathbf{U}^j v$ containing v is one-dimensional, there exists a unique irreducible submodule $L \subset \mathbf{U}^j v$ containing v . This shows that $\mathbf{U}^j v = L$ is irreducible. This proves the corollary. \square

Theorem 4.4.5. *Let $M \in \mathcal{O}_{\text{int}}^J$. Irreducible decomposition of M is unique in the following sense. If we have two irreducible decompositions $M = \bigoplus_{j \in J} L_j = \bigoplus_{k \in K} L^k$ for some index sets J and K , then there exists a bijection $\phi : J \rightarrow K$ such that $L_j \simeq L^{\phi(j)}$ for all $j \in J$. Moreover, for each $j \in J$, the number of $j' \in J$ such that $L_{j'} \simeq L_j$ is finite.*

Proof. Set $H := \{v \in M \mid \mathbf{U}_{>0}^J v = 0\}$. Suppose that we have an irreducible decomposition $M = \bigoplus_{j \in J} L_j$ of M for some index set J . Then, we obtain a highest weight vector $v_j \in L_j$ with highest weight $(\lambda_j, \mathbf{H}_j)$ for each $j \in J$. Clearly, $\{v_j \mid j \in J\}$ is a basis of H consisting of simultaneous eigenvectors of h_i , $i \in \mathbb{I}$. This shows that each irreducible decomposition of M depends only on the choice of a base of $H_{\lambda, \mathbf{H}} := \{v \in H \cap M_\lambda \mid h_i v = H_i v \text{ for } i \in \mathbb{I}\}$ for each $\lambda \in \Lambda^J$, $\mathbf{H} \in \mathbb{Q}(p, q)^r$. Since each weight space M_λ is finite-dimensional, so is $H_{\lambda, \mathbf{H}}$ for all $\mathbf{H} \in \mathbb{Q}(p, q)^r$. Therefore, the number of $j \in J$ such that $L_j \simeq L(\lambda; \mathbf{H})$ is equal to $\dim H_{\lambda, \mathbf{H}}$ for all $\lambda \in \Lambda^J$, $\mathbf{H} \in \mathbb{Q}(p, q)^r$. This proves the theorem. \square

5. QUASI- J -CRYSTAL BASES

5.1. Quasi- J -crystal bases. The $\mathbf{U}^J = \mathbf{U}_r^J$ has $r - 1$ \mathfrak{sl}_2 -triples: (f_i, k_i, e_i) for $i = 2, \dots, r$. Hence, one can define Kashiwara operators, \tilde{f}_i and \tilde{e}_i , in the same way as in the crystal basis theory for quantum groups. Also, by the case $r = 1$, one can define Kashiwara operators, \tilde{f}_1 and \tilde{e}_1 . Let us give the precise definition of these operators.

Definition 5.1.1. Let M be a \mathbf{U}^J -module, $\lambda \in \Lambda^J$, and $m \in M_\lambda$. For each $i \in \mathbb{I}^J$, there exist $m_j \in M_{\lambda + j\gamma_i}$, $j = 0, 1, \dots, N$, uniquely for some N such that

$$e_i m_j = 0, \quad e_i f_i m_j \in \mathbb{Q}(p, q) m_j, \quad m = \sum_{j=0}^N f_i^{(j)} m_j.$$

Using this expression, we define $\tilde{f}_i(m)$ and $\tilde{e}_i(m)$ by:

$$\tilde{f}_i(m) = \sum_{j=0}^N f_i^{(j+1)} m_j, \quad \tilde{e}_i(m) = \sum_{j=1}^N f_i^{(j-1)} m_j.$$

Set $\mathbf{A}_0 := \{f/g \in \mathbb{Q}(p, q) \mid f, g \in p\mathbb{Q}[p, q, q^{-1}] + \mathbb{Q}[q], g \notin p\mathbb{Q}[p, q, q^{-1}] + q\mathbb{Q}[q]\}$; namely, \mathbf{A}_0 consists of all those $h \in \mathbb{Q}(p, q)$ for which $\lim_{q \rightarrow 0}(\lim_{p \rightarrow 0} h)$ exists.

Definition 5.1.2. Let M be a \mathbf{U}^J -module and \mathcal{L} an \mathbf{A}_0 -submodule of M . We say that \mathcal{L} is a quasi- J -crystal lattice of M if

- (qL 1) $\mathbb{Q}(p, q) \otimes_{\mathbf{A}_0} \mathcal{L} = M$,
- (qL 2) $\mathcal{L} = \bigoplus_{\lambda \in \Lambda^J} \mathcal{L}_\lambda$, where $\mathcal{L}_\lambda := \mathcal{L} \cap M_\lambda$,
- (qL 3) $\tilde{f}_i(\mathcal{L}) \subset \mathcal{L}$ and $\tilde{e}_i(\mathcal{L}) \subset \mathcal{L}$ for all $i \in \mathbb{I}^J$.

If \mathcal{L} is a quasi- J -crystal lattice of M , then Kashiwara operators induce \mathbb{Q} -linear maps, denoted by the same symbols, on $\mathcal{L}/q\mathcal{L}$.

Definition 5.1.3. Let M be a \mathbf{U}^J -module, \mathcal{L} an \mathbf{A}_0 -submodule of M , and \mathcal{B} a subset of $\mathcal{L}/q\mathcal{L}$. We say that $(\mathcal{L}, \mathcal{B})$ is a quasi- J -crystal basis if

- (qB 1) \mathcal{L} is a quasi- J -crystal lattice of M ,
- (qB 2) \mathcal{B} is a \mathbb{Q} -basis of $\mathcal{L}/q\mathcal{L}$,
- (qB 3) $\tilde{\mathcal{B}} = \bigsqcup_{\lambda \in \Lambda^J} \mathcal{B}_\lambda$, where $\mathcal{B}_\lambda := \mathcal{B} \cap (\mathcal{L}_\lambda/q\mathcal{L}_\lambda)$,
- (qB 4) $\tilde{f}_i(\mathcal{B}) \subset \mathcal{B} \sqcup \{0\}$ and $\tilde{e}_i(\mathcal{B}) \subset \mathcal{B} \sqcup \{0\}$ for all $i \in \mathbb{I}^J$,
- (qB 5) for each $b, b' \in \mathcal{B}$ and $i \in \mathbb{I}^J$, one has $\tilde{f}_i(b) = b'$ if and only if $b = \tilde{e}_i(b')$.

Definition 5.1.4. For a quasi- j -crystal basis $(\mathcal{L}, \mathcal{B})$ and $i \in \mathbb{I}^j$, we define three maps $\varphi_i : \mathcal{B} \rightarrow \mathbb{Z}_{\geq 0}$, $\varepsilon_i : \mathcal{B} \rightarrow \mathbb{Z}_{\geq 0}$, and $\text{wt} : \mathcal{B} \rightarrow \Lambda^j$ by

$$\begin{aligned}\varphi_i(b) &:= \max\{n \mid \tilde{f}_i^n(b) \neq 0\}, \\ \varepsilon_i(b) &:= \max\{n \mid \tilde{e}_i^n(b) \neq 0\}, \\ \text{wt}(b) &:= \lambda \text{ if } b \in \mathcal{B}_\lambda.\end{aligned}$$

Example 5.1.5. Let $r = 1$. For each $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq 0}$, the irreducible \mathbf{U}_1^j -module $L(a; b)$ has the following quasi- j -crystal basis. Let $\mathcal{L}(a; b)$ denote the \mathbf{A}_0 -lattice spanned by $\{f_1^{(n)}v \mid 0 \leq n \leq b\}$, and set $\mathcal{B}(a; b) := \{f_1^{(n)}v + q\mathcal{L} \mid 0 \leq n \leq b\}$. Then, the Kashiwara operators \tilde{f}_1 and \tilde{e}_1 act on $\mathcal{B}(a; b)$ by:

$$\tilde{f}_1(f_1^{(n)}v + q\mathcal{L}) = f_1^{(n+1)}v + q\mathcal{L}, \quad \tilde{e}_1(f_1^{(n)}v + q\mathcal{L}) = f_1^{(n-1)}v + q\mathcal{L}.$$

In addition, one has $\varphi_1(f_1^{(n)}v + q\mathcal{L}) = b - n$, $\varepsilon_1(f_1^{(n)}v + q\mathcal{L}) = n$, and $\text{wt}(f_1^{(n)}v + q\mathcal{L}) = (a - 3n)\delta_1$.

Definition 5.1.6. Let M be a \mathbf{U}^j -module and $(\mathcal{L}, \mathcal{B})$ a quasi- j -crystal basis of M . The quasi- j -crystal graph associated with $(\mathcal{L}, \mathcal{B})$ is the colored directed graph with vertex set \mathcal{B} and edges $b \xrightarrow{i} b'$, where $b, b' \in \mathcal{B}$, $i \in \mathbb{I}^j$ are such that $\tilde{f}_i b = b'$.

Note that a quasi- j -crystal graph of an irreducible module is usually disconnected unless $r = 1$.

5.2. Tensor product rule. Recall that \mathbf{U}^j is a right coideal of \mathbf{U} , i.e., $\Delta(\mathbf{U}^j) \subset \mathbf{U}^j \otimes \mathbf{U}$. Hence, we are interested in the \mathbf{U}^j -module structure of the tensor product of a \mathbf{U}^j -module and a \mathbf{U} -module. Let V_r denote the vector representation of \mathbf{U} . It is spanned by $\{u_n \mid -r \leq n \leq r\}$, and is equipped with a \mathbf{U} -module structure by:

$$F_j u_i = \delta_{j-\frac{1}{2}, i} u_{i+1}, \quad E_j u_i = \delta_{j+\frac{1}{2}, i} u_{i-1}, \quad K_j u_i = q^{(\alpha_j, \epsilon_i)} u_i.$$

If we set $\mathcal{L}_r := \bigoplus_{n=-r}^r \mathbf{A}_0 u_n$, $\mathcal{B}_r := \{u_n + q\mathcal{L}_r \mid -r \leq n \leq r\}$, then, $(\mathcal{L}_r, \mathcal{B}_r)$ is an ordinary crystal basis of V_r .

When we consider ordinary crystal bases, Kashiwara operators acting on them are denoted by capital letters \tilde{E}_i and \tilde{F}_i , $i \in \mathbb{I}$, while those for quasi- j -crystal bases are denoted by lowercase letters \tilde{e}_i and \tilde{f}_i , $i \in \mathbb{I}^j$.

We first consider the case $r = 1$.

Proposition 5.2.1. Let $a \in \mathbb{Z}$, $b \in \mathbb{Z}_{\geq 0}$. Then we have an isomorphism

$$L(a; b) \otimes V_1 \simeq L(a+2; b+1) \oplus L(a-1; b) \oplus L(a-1; b-1)$$

of \mathbf{U}_1^j -modules. Moreover, $(\mathcal{L}(a; b) \otimes \mathcal{L}_1, \mathcal{B}(a; b) \otimes \mathcal{B}_1)$ is a quasi- j -crystal basis of $L(a; b) \otimes V_1$.

Proof. Let $v \in L(a; b)$ be a highest weight vector, and set

$$\begin{aligned}v[0] &:= v \otimes u_0, \\ v[1] &:= v \otimes u_1 - \frac{q^{-b+1}(q-q^{-1})}{\{a-b-1\}} f_1 v \otimes u_0 - pq^{a-2b} v \otimes u_{-1}, \\ v[-1] &:= f_1 v \otimes u_0 - q^b [b] v \otimes u_{-1} - pq^{a-b-2} [b] v \otimes u_1.\end{aligned}$$

Then, by direct calculation, we obtain

$$\begin{aligned}h_1 v[0] &= [b+1]\{(a+2) - (b+1) - 1\} v[0], \\ h_1 v[1] &= [b]\{(a-1) - b - 1\} v[1], \\ h_1 v[-1] &= [b-1]\{(a-1) - (b-1) - 1\} v[-1].\end{aligned}$$

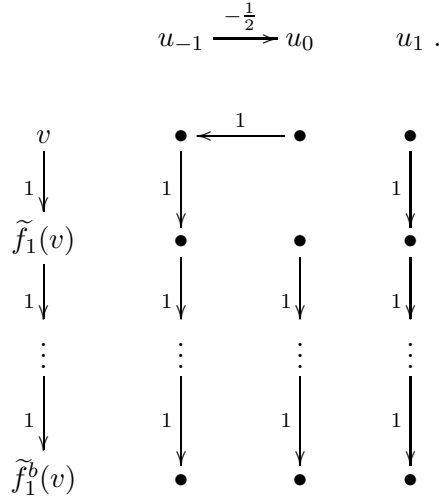
These equations, together with Corollary 3.2.4 and Theorem 3.1.6, show that $\mathbf{U}_1^j v[0] \simeq L(a+2; b+1)$, $\mathbf{U}_1^j v[1] \simeq L(a-1; b)$, and $\mathbf{U}_1^j v[-1] \simeq L(a-1; b-1)$. Since $\dim(L(a; b) \otimes V_1) = 3b =$

$(b+1)+b+(b-1) = \sum_{k=-1}^1 \dim \mathbf{U}_1^j v[\overline{k}]$, we see that $L(a; b) \otimes V_1 = \mathbf{U}_1^j v[\overline{0}] \oplus \mathbf{U}_1^j v[\overline{-1}] \oplus \mathbf{U}_1^j v[\overline{1}]$. Also, we calculate as:

$$\begin{aligned}
f_1^{(n)}(v[\overline{0}]) &= f_1^{(n-1)}v \otimes u_{-1} + q^n f_1^{(n)}v \otimes u_0 + pq^{a-n+1} f_1^{(n-1)}v \otimes u_1 \\
&\in \begin{cases} v \otimes u_0 + q\mathcal{L}(a; b) \otimes \mathcal{L}_1 & \text{if } n = 0, \\ f_1^{(n-1)}v \otimes u_{-1} + q\mathcal{L}(a; b) \otimes \mathcal{L}_1 & \text{if } 0 \leq n \leq b+1, \end{cases} \\
f_1^{(n)}(v[\overline{1}]) &= \frac{q^{-n}\{a-b-n-1\}}{\{a-b-1\}} f_1^{(n)}v \otimes u_1 - \frac{q^{-b+n+1}(q^{n+1} - q^{-n-1})}{\{a-b-1\}} f_1^{(n+1)}v \otimes u_0 \\
&\quad - \frac{pq^{a-2b}\{a-b-n-1\}}{\{a-b-1\}} f_1^{(n)}v \otimes u_{-1} \\
&\in f_1^{(n)}v \otimes u_1 + q\mathcal{L}(a; b) \otimes \mathcal{L}_1 \quad \text{if } 0 \leq n \leq b, \\
f_1^{(n)}(v[\overline{-1}]) &= q^n[n+1]f_1^{(n+1)}v \otimes u_0 - q^b[b-n]f_1^{(n)}v \otimes u_{-1} - pq^{a-b-n-2}[b-n]f_1^{(n)}v \otimes u_1 \\
&\in f_1^{(n+1)}v \otimes u_0 + q\mathcal{L}(a; b) \otimes \mathcal{L}_1 \quad \text{if } 0 \leq n \leq b-1.
\end{aligned}$$

Since $\tilde{f}_1^n(v[\overline{k}]) = f_1^{(n)}(v[\overline{k}])$, $k \in \{0, \pm 1\}$, these equations imply that the \mathbf{A}_0 -span of $\{\tilde{f}_1^n(v[\overline{k}]) \mid k \in \{0, \pm 1\}, n \in \mathbb{Z}_{\geq 0}\}$ coincides with $\mathcal{L}(a; b) \otimes \mathcal{L}_1$, and that $\{\tilde{f}_1^n(v[\overline{k}]) + q\mathcal{L}(a; b) \otimes \mathcal{L}_1 \mid k \in \{0, \pm 1\}, n \in \mathbb{Z}_{\geq 0}\} \setminus \{0\}$ is identical to $\mathcal{B}(a; b) \otimes \mathcal{B}_1$. Now, it is easy to verify that $(\mathcal{L}(a; b) \otimes \mathcal{L}_1, \mathcal{B}(a; b) \otimes \mathcal{B}_1)$ is a quasi- j -crystal basis of $L(a; b) \otimes V_1$. This proves the proposition. \square

We give the quasi- j -crystal graph of $\mathcal{B}(a; b) \otimes \mathcal{B}_1$:



Let $N \in \mathbb{N}$. Applying the above proposition repeatedly, we see that the tensor product module $V_1^{\otimes N}$ has a quasi- j -crystal basis $(\mathcal{L}_1^{\otimes N}, \mathcal{B}_1^{\otimes N})$; we denote $u_{i_1} \otimes \cdots \otimes u_{i_N} + q\mathcal{L}_1^{\otimes N} \in \mathcal{B}_1^{\otimes N}$, $i_1, \dots, i_N \in \{-1, 0, 1\}$, by (i_1, \dots, i_N) . For each $\mathbf{s} = (s_1, \dots, s_N) \in \mathcal{B}_1^{\otimes N}$, let us describe $\tilde{f}_1(\mathbf{s})$ and $\tilde{e}_1(\mathbf{s})$ explicitly. First, ignore all the s_j 's such that $s_j = 1$. Next, delete all the adjacent ordered pairs $(-1, 0)$, and repeat this until there are no such pairs. The resulting sequence $\mathbf{s}_{-\frac{1}{2}}$ (the subscript is for later use) is of the form $(0, \dots, 0, -1, \dots, -1)$. Then, $\tilde{f}_1(\mathbf{s})$ (resp., $\tilde{e}_1(\mathbf{s})$) is obtained from \mathbf{s} by replacing the rightmost 0 in $\mathbf{s}_{-\frac{1}{2}}$ with -1 (resp., the leftmost -1 in $\mathbf{s}_{-\frac{1}{2}}$ with 0); if this is impossible, then $\tilde{f}_1(\mathbf{s})$ (resp., $\tilde{e}_1(\mathbf{s})$) equals 0. Namely, $\tilde{f}_1(\mathbf{s}) = \tilde{E}_{-\frac{1}{2}}(\mathbf{s})$ and $\tilde{e}_1(\mathbf{s}) = \tilde{F}_{-\frac{1}{2}}(\mathbf{s})$.

For example, let $\mathbf{s} = (0, 0, -1, 1, 1, 1, -1, 0, 1, 0, 0, 0, -1)$. Then,

$$\begin{aligned}\mathbf{s}_{-\frac{1}{2}} &= (0, 0, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, 0, 0, -1), \\ \tilde{f}_1(\mathbf{s}) &= (0, 0, -1, 1, 1, 1, -1, 0, 1, 0, 0, -1, -1), \\ \tilde{e}_1(\mathbf{s}) &= (0, 0, -1, 1, 1, 1, -1, 0, 1, 0, 0, 0, 0).\end{aligned}$$

More generally, we obtain the following theorem. As in the ordinary crystal basis theory, the proof is given by embedding the crystal basis of a \mathbf{U}_3 -module into $(\mathcal{L}_1^{\otimes N}, \mathcal{B}_1^{\otimes N})$ for a sufficiently large N .

Theorem 5.2.2. *Let M be a \mathbf{U}_1^j -module having a quasi- j -crystal basis $(\mathcal{L}, \mathcal{B})$, and N a \mathbf{U}_3 -module having a crystal basis $(\mathcal{L}', \mathcal{B}')$. Then, $M \otimes N$ has a quasi- j -crystal basis $(\mathcal{L} \otimes \mathcal{L}', \mathcal{B} \otimes \mathcal{B}')$, on which the Kashiwara operators act as follows:*

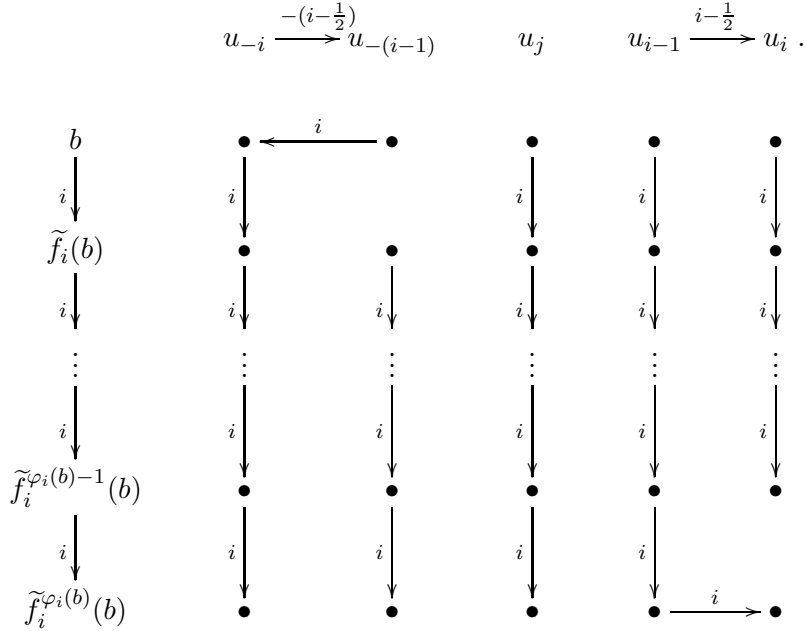
$$\begin{aligned}\tilde{f}_1(b \otimes b') &= \begin{cases} b \otimes \tilde{E}_{-\frac{1}{2}}(b') & \text{if } \varepsilon_1(b) < \varepsilon_{-\frac{1}{2}}(b'), \\ \tilde{f}_1(b) \otimes b' & \text{if } \varepsilon_1(b) \geq \varepsilon_{-\frac{1}{2}}(b'), \end{cases} \\ \tilde{e}_1(b \otimes b') &= \begin{cases} b \otimes \tilde{F}_{-\frac{1}{2}}(b') & \text{if } \varepsilon_1(b) \leq \varepsilon_{-\frac{1}{2}}(b'), \\ \tilde{e}_1(b) \otimes b' & \text{if } \varepsilon_1(b) > \varepsilon_{-\frac{1}{2}}(b'). \end{cases}\end{aligned}$$

Now, we turn to the case of a general r . Recall that Kashiwara operators \tilde{f}_i and \tilde{e}_i for $i \neq 1$ are defined by means of the \mathfrak{sl}_2 -triple (f_i, k_i, e_i) . Therefore, the next proposition follows from a standard argument; see, for example, [HK02, Section 4.4].

Proposition 5.2.3. *Let M be a \mathbf{U}^j -module having a quasi- j -crystal basis $(\mathcal{L}, \mathcal{B})$. Then $(\mathcal{L} \otimes \mathcal{L}_r, \mathcal{B} \otimes \mathcal{B}_r)$ is a quasi- j -crystal basis of $M \otimes V_r$, on which the Kashiwara operators act as follows: \tilde{f}_1 and \tilde{e}_1 acts as described in Theorem 5.2.2; for $i \in \mathbb{I} \setminus \{1\}$, $b \in \mathcal{B}$, $j \in \{-r, -r+1, \dots, r\}$,*

$$\begin{aligned}\tilde{f}_i(b \otimes u_j) &= \begin{cases} 0 & \text{if } j = i \text{ and } \tilde{f}_i^2(b) = 0, \\ b \otimes u_i & \text{if } j = i-1 \text{ and } \tilde{f}_i(b) = 0, \\ b \otimes u_{-i} & \text{if } j = -(i-1) \text{ and } \tilde{e}_i(b) = 0, \\ \tilde{f}_i(b) \otimes u_j & \text{otherwise,} \end{cases} \\ \tilde{e}_i(b \otimes u_j) &= \begin{cases} b \otimes u_{i-1} & \text{if } j = i \text{ and } \tilde{f}_i(b) = 0, \\ 0 & \text{if } j = -(i-1) \text{ and } \tilde{e}_i^2(b) = 0, \\ b \otimes u_{-(i-1)} & \text{if } j = -i \text{ and } \tilde{e}_i(b) = 0, \\ \tilde{e}_i(b) \otimes u_j & \text{otherwise.} \end{cases}\end{aligned}$$

The action of \tilde{f}_i for $i \neq 1$ is visualized as:



From Proposition 5.2.3, we see that $V_r^{\otimes N}$ has a quasi- j -crystal basis $(\mathcal{L}_r^{\otimes N}, \mathcal{B}_r^{\otimes N})$; we denote $u_{i_1} \otimes \cdots \otimes u_{i_N} + q\mathcal{L}_r^{\otimes N} \in \mathcal{B}_r^{\otimes N}$, $i_1, \dots, i_N \in \{-r, \dots, r\}$, by (i_1, \dots, i_N) . Before describing its quasi- j -crystal structure, recall the ordinary crystal structure of $\mathcal{B}_r^{\otimes N}$. Let $\mathbf{s} = (s_1, \dots, s_N) \in \mathcal{B}_r^{\otimes N}$ and $i \in \mathbb{I}$. First, ignore all the s_j 's such that $s_j \neq i \pm \frac{1}{2}$. Next, delete all the adjacent ordered pairs $(i - \frac{1}{2}, i + \frac{1}{2})$, and repeat this until there are no such pairs. The resulting sequence is of the form $(i + \frac{1}{2}, \dots, i + \frac{1}{2}, i - \frac{1}{2}, \dots, i - \frac{1}{2})$; we denote this sequence by \mathbf{s}_i . Then, $\tilde{F}_i(\mathbf{s})$ is obtained from \mathbf{s} by replacing the leftmost $i - \frac{1}{2}$ in \mathbf{s}_i with $i + \frac{1}{2}$; if this is impossible, then $\tilde{F}_i(\mathbf{s}) = 0$. Also, $\tilde{E}_i(\mathbf{s})$ is obtained from \mathbf{s} by replacing the rightmost $i + \frac{1}{2}$ in \mathbf{s}_i with $i - \frac{1}{2}$; if this is impossible, then $\tilde{E}_i(\mathbf{s}) = 0$. Note that \mathbf{s}_i consists of $\varepsilon_i(\mathbf{s})$ $(i + \frac{1}{2})$'s and $\varphi_i(\mathbf{s})$ $(i - \frac{1}{2})$'s:

$$\mathbf{s}_i = \underbrace{\left(i + \frac{1}{2}, \dots, i + \frac{1}{2}\right)}_{\varepsilon_i(\mathbf{s})} \underbrace{\left(i - \frac{1}{2}, \dots, i - \frac{1}{2}\right)}_{\varphi_i(\mathbf{s})}.$$

From the consideration above, we can describe the quasi- j -crystal structure of $\mathcal{B}_r^{\otimes N}$ as follows.

Proposition 5.2.4. *Let \mathbf{s} be as before and $i \in \mathbb{I} \setminus \{1\}$. First, consider the concatenated sequence $(\mathbf{s}_{-(i-\frac{1}{2})}^{\text{rev}}, \mathbf{s}_{i-\frac{1}{2}})$, where $\mathbf{s}_{-(i-\frac{1}{2})}^{\text{rev}}$ is the sequence obtained by reversing the order of $\mathbf{s}_{-(i-\frac{1}{2})}$; we denote this sequence by \mathbf{s}_i , i.e.,*

$$\mathbf{s}_i = \underbrace{(-i, \dots, -i)}_{\varphi_{-(i-\frac{1}{2})}(\mathbf{s})} \underbrace{(-(i-1), \dots, -(i-1))}_{\varepsilon_{-(i-\frac{1}{2})}(\mathbf{s})} \underbrace{(i, \dots, i)}_{\varepsilon_{i-\frac{1}{2}}(\mathbf{s})} \underbrace{(i-1, \dots, i-1)}_{\varphi_{i-\frac{1}{2}}(\mathbf{s})}.$$

Next, delete all the adjacent ordered pairs $(-(i-1), i)$, and repeat this until there are no such pairs. Then, $\tilde{f}_i(\mathbf{s})$ is obtained from \mathbf{s} by replacing the leftmost $-(i-1)$ in \mathbf{s}_i with $-i$ (resp., the leftmost $i-1$ in \mathbf{s}_i with i) if $-(i-1) \in \mathbf{s}_i$ (resp., $-(i-1) \notin \mathbf{s}_i$); if this is impossible, then $\tilde{f}_i(\mathbf{s}) = 0$. Also, $\tilde{e}_i(\mathbf{s})$ is obtained from \mathbf{s} by replacing the rightmost i in \mathbf{s}_i with $i-1$ (resp., the rightmost $-i$ in \mathbf{s}_i with $-(i-1)$) if $i \in \mathbf{s}_i$ (resp., $i \notin \mathbf{s}_i$); if this is impossible, then $\tilde{e}_i(\mathbf{s}) = 0$.

Namely,

$$\begin{aligned}\tilde{f}_1(\mathbf{s}) &= \tilde{E}_{-\frac{1}{2}}(\mathbf{s}), & \tilde{e}_1(\mathbf{s}) &= \tilde{F}_{-\frac{1}{2}}(\mathbf{s}), \\ \tilde{f}_i(\mathbf{s}) &= \begin{cases} \tilde{E}_{-(i-\frac{1}{2})}(\mathbf{s}) & \text{if } \varepsilon_{i-\frac{1}{2}}(\mathbf{s}) < \varepsilon_{-(i-\frac{1}{2})}(\mathbf{s}), \\ \tilde{F}_{i-\frac{1}{2}}(\mathbf{s}) & \text{if } \varepsilon_{i-\frac{1}{2}}(\mathbf{s}) \geq \varepsilon_{-(i-\frac{1}{2})}(\mathbf{s}), \end{cases} \\ \tilde{e}_i(\mathbf{s}) &= \begin{cases} \tilde{F}_{-(i-\frac{1}{2})}(\mathbf{s}) & \text{if } \varepsilon_{i-\frac{1}{2}}(\mathbf{s}) \leq \varepsilon_{-(i-\frac{1}{2})}(\mathbf{s}), \\ \tilde{E}_{i-\frac{1}{2}}(\mathbf{s}) & \text{if } \varepsilon_{i-\frac{1}{2}}(\mathbf{s}) > \varepsilon_{-(i-\frac{1}{2})}(\mathbf{s}). \end{cases}\end{aligned}$$

Proof. The proof proceeds by induction on N by means of Proposition 5.2.3. \square

Now, we are ready to generalize Proposition 5.2.3. The following theorem describes the tensor product rule for the Kashiwara operators \tilde{f} 's and \tilde{e} 's in full generality. The proof is given by embedding the crystal basis of a \mathbf{U} -module into $(\mathcal{L}_r^{\otimes N}, \mathcal{B}_r^{\otimes N})$ for a sufficiently large N .

Theorem 5.2.5. *Let M be a \mathbf{U}_r^j -module having a quasi- j -crystal basis $(\mathcal{L}, \mathcal{B})$, and N a \mathbf{U}_{2r+1} -module having a crystal basis $(\mathcal{L}', \mathcal{B}')$. Then, $M \otimes N$ has a quasi- j -crystal basis $(\mathcal{L} \otimes \mathcal{L}', \mathcal{B} \otimes \mathcal{B}')$, on which the Kashiwara operators act as follows: for $b \in \mathcal{B}$ and $b' \in \mathcal{B}'$,*

$$\begin{aligned}\tilde{f}_1(b \otimes b') &= \begin{cases} b \otimes \tilde{E}_{-\frac{1}{2}}(b') & \text{if } \varepsilon_1(b) < \varepsilon_{-\frac{1}{2}}(b'), \\ \tilde{f}_1(b) \otimes b' & \text{if } \varepsilon_1(b) \geq \varepsilon_{-\frac{1}{2}}(b'), \end{cases} \\ \tilde{e}_1(b \otimes b') &= \begin{cases} b \otimes \tilde{F}_{-\frac{1}{2}}(b') & \text{if } \varepsilon_1(b) \leq \varepsilon_{-\frac{1}{2}}(b'), \\ \tilde{e}_1(b) \otimes b' & \text{if } \varepsilon_1(b) > \varepsilon_{-\frac{1}{2}}(b'), \end{cases} \\ \tilde{f}_i(b \otimes b') &= \begin{cases} b \otimes \tilde{E}_{-(i-\frac{1}{2})}(b') & \text{if } \varepsilon_{i-\frac{1}{2}}(b') < \varphi_i(b) \text{ and } \varepsilon_i(b) < \varepsilon_{-(i-\frac{1}{2})}(b'), \text{ or} \\ & \text{if } \varepsilon_{i-\frac{1}{2}}(b') \geq \varphi_i(b) \text{ and } \varepsilon_i(b) + \varepsilon_{i-\frac{1}{2}}(b') - \varphi_i(b) < \varepsilon_{-(i-\frac{1}{2})}(b'), \\ \tilde{f}_i(b) \otimes b' & \text{if } \varepsilon_{i-\frac{1}{2}}(b') < \varphi_i(b) \text{ and } \varepsilon_i(b) \geq \varepsilon_{-(i-\frac{1}{2})}(b'), \\ b \otimes \tilde{F}_{i-\frac{1}{2}}(b') & \text{if } \varepsilon_{i-\frac{1}{2}}(b') \geq \varphi_i(b) \text{ and } \varepsilon_i(b) + \varepsilon_{i-\frac{1}{2}}(b') - \varphi_i(b) \geq \varepsilon_{-(i-\frac{1}{2})}(b'), \end{cases} \\ \tilde{e}_i(b \otimes b') &= \begin{cases} b \otimes \tilde{F}_{-(i-\frac{1}{2})}(b') & \text{if } \varepsilon_{i-\frac{1}{2}}(b') \leq \varphi_i(b) \text{ and } \varepsilon_i(b) \leq \varepsilon_{-(i-\frac{1}{2})}(b'), \text{ or} \\ & \text{if } \varepsilon_{i-\frac{1}{2}}(b') > \varphi_i(b) \text{ and } \varepsilon_i(b) + \varepsilon_{i-\frac{1}{2}}(b') - \varphi_i(b) \leq \varepsilon_{-(i-\frac{1}{2})}(b'), \\ \tilde{e}_i(b) \otimes b' & \text{if } \varepsilon_{i-\frac{1}{2}}(b') \leq \varphi_i(b) \text{ and } \varepsilon_i(b) > \varepsilon_{-(i-\frac{1}{2})}(b'), \\ b \otimes \tilde{E}_{i-\frac{1}{2}}(b') & \text{if } \varepsilon_{i-\frac{1}{2}}(b') > \varphi_i(b) \text{ and } \varepsilon_i(b) + \varepsilon_{i-\frac{1}{2}}(b') - \varphi_i(b) > \varepsilon_{-(i-\frac{1}{2})}(b'). \end{cases}\end{aligned}$$

6. THE CASE $r = 2$

6.1. Quasi- j -crystal bases of irreducible highest weight modules. Throughout this subsection, we fix a \mathbf{U}_2^j -module $M \in \mathcal{O}_{\text{int}}^j$. Recall from the case $r = 1$ that M is decomposed as:

$$\begin{aligned}M &= \bigoplus_{\substack{a \in \mathbb{Z} \\ b, n \in \mathbb{Z}_{\geq 0}}} M_{a, b, n}, \\ M_{a, b, 0} &= \{u \in M \mid e_1 u = 0, \ k_1 u = q^a u, \ h_1 u = [b]\{a - b - 1\}u\}, \\ M_{a, b, n} &= f_1^{(n)}(M_{a, b, 0}).\end{aligned}$$

While the representation theory of \mathbf{U}_1^j is similar to that of \mathbf{U}_2 , the representation theory of \mathbf{U}_2^j is much more difficult than that of \mathbf{U}_3 . The main difficulty comes from the fact that $f_2 m$ is not necessarily an eigenvector of h_1 even if m is so. Hence, we need to investigate the action of f_2 on m carefully.

Recall that $h_1 = [e_1, f_1]_1$ and $h_2 = \tau_2(h_1)$. Set $f'_2 := q^{-2}[e_1, [f_1, f_2]_1]_1 - p^{-1}q^{-1}f_2k_1^{-1}$. For each $a \in \mathbb{Z}$ and $b, n \in \mathbb{Z}_{\geq 0}$, we define $f'_{2,i}(a, b, n) \in \mathbf{U}^j$, $i = 1, 2, 3$, by

$$\begin{aligned} f'_{2,1}(a, b, n) &:= q^{b-n-1}\overline{f'_2} + (pq^{a-b} - p^{-1}q^{-a+b})f_2 - q^{-b+n+1}f'_2, \\ f'_{2,2}(a, b, n) &:= pq^{a-b-n-2}\overline{f'_2} - (q^{b+1} + q^{-b-1})f_2 + p^{-1}q^{-a+b+n+2}f'_2, \\ f'_{2,3}(a, b, n) &:= q^{-n-2}\overline{f'_2} + (pq^{a-2b-1} - p^{-1}q^{-a+2b+1})f_2 - q^{n+2}f'_2. \end{aligned}$$

Also, we define three linear maps $f'_{2,i}$, $i = 1, 2, 3$, by

$$f'_{2,i}(m) := f'_{2,i}(a, b, n)m \quad \text{for } m \in M_{a,b,n}.$$

Proposition 6.1.1. *Let $a \in \mathbb{Z}$, $b, n \in \mathbb{Z}_{\geq 0}$, and $m \in M_{a,b,n}$. Then, we have*

$$f'_{2,1}(m) \in M_{a+1,b+1,n}, \quad f'_{2,2}(m) \in M_{a+1,b,n}, \quad f'_{2,3}(m) \in M_{a-2,b-1,n-1}.$$

Proof. See Appendix B.1 □

Proposition 6.1.2. *The linear maps $f'_{2,i}$, $i = 1, 2, 3$, commute with each other.*

Proof. See Appendix B.2 □

We normalize $f'_{2,i}$ as follows:

$$\begin{aligned} f_{2,1}(a, b, n) &:= \frac{1}{(q^{b+1} - q^{-b-1})\{a - 2b - 1\}} f'_{2,1}(a, b, n), \\ f_{2,2}(a, b, n) &:= -\frac{1}{\{a - b\}\{a - 2b - 1\}} f'_{2,2}(a, b, n), \\ f_{2,3}(a, b, n) &:= -\frac{1}{(q^{b+1} - q^{-b-1})\{a - b\}} f'_{2,3}(a, b, n), \end{aligned}$$

and define linear maps $f_{2,i}$, $i = 1, 2, 3$, by $f_{2,i}(m) = f_{2,i}(a, b, n)m$ for $m \in M_{a,b,n}$. Then, for each $m \in M_{a,b,n}$, we have $f_2m = (f_{2,1} + f_{2,2} + f_{2,3})m$. Thanks to this equality and Proposition 6.1.1, in order to compute $f_{2,i}(m)$, it is enough to decompose f_2m into three h_1 -eigenvectors with distinct eigenvalues. The computation becomes easier when $n = 0$ since in this case, $f_{2,3}(m) = 0$. Also, it follows that $f_2m \in M_{a+1,b+1,0} \oplus M_{a+1,b,0}$ for $m \in M_{a,b,0}$. Repeating this, we have

$$(9) \quad f_2^{(l)}m \in \bigoplus_{k=0}^l M_{a+l,b+k,0} \quad \text{for } l \in \mathbb{Z}_{\geq 0}, \quad m \in M_{a,b,0}.$$

Let $L \in \mathcal{O}_{\text{int}}^j$ be an irreducible \mathbf{U}_2^j -module. By the first half of the proof of Theorem 4.3.5, L is isomorphic to $L(a_1\delta_1 + a_2\delta_2; H_1, H_2)$ for some $a_1 \in \mathbb{Z}$, $a_2 \in \mathbb{Z}_{\geq 0}$, and $H_1, H_2 \in \mathbb{Q}(p, q)$. Moreover, $H_1 = [b_1]\{a_1 - b_1 - 1\}$ for some $b_1 \in \mathbb{Z}_{\geq 0}$. As we announced in the proof of Theorem 4.3.5, we show that $H_2 = [b_1 + b_2]\{a_1 + a_2 - (b_1 + b_2) - 1\}$ for some $b_2 \in \mathbb{Z}_{\geq 0}$. If $v \in L$ is a highest weight vector, then $\tau_2^{-1}(v)$ is a \mathbf{U}_1^j -highest weight vector. Hence, we have

$$h_2v = \tau_2(h_1\tau_2^{-1}(v)) = [b]\{a_1 + a_2 - b - 1\}v$$

for some $b \in \mathbb{Z}_{\geq 0}$. This implies that $f_2^{(a_2)}v = \tau_2^{-1}(v) \in L_{a_1+a_2,b,0}$. Also, we have $f_2^{(a_2)}v \in \bigoplus_{k=0}^{a_2} L_{a_1+a_2,b_1+k,0}$ by equation (9). Therefore, we deduce that $b_1 \leq b \leq b_1 + a_2$, and hence $b = b_1 + b_2$ for some $0 \leq b_2 \leq a_2$. This completes the proof of Theorem 4.3.5.

By the above, we may assume that $L = L(a_1, a_2; b_1, b_2)$ for some $a_1 \in \mathbb{Z}$, $b_1 \in \mathbb{Z}_{\geq 0}$, and $0 \leq b_2 \leq a_2$. We define two linear operators $\tilde{f}_{2'}$ and $\tilde{e}_{2'}$ on L as follows. First, for $c \in \mathbb{Z}_{\geq 0}$, we set

$$\tilde{f}_{2'}^c(v) := q^{-cb_2}f_{2,2}^{(c)}v,$$

and

$$\tilde{f}_{2'}(\tilde{f}_{2'}^c(v)) = \tilde{f}_{2'}^{c+1}(v), \quad \tilde{e}_{2'}(\tilde{f}_{2'}^c(v)) = \tilde{f}_{2'}^{c-1}(v) \quad \text{if } \tilde{f}_{2'}^c(v) \neq 0.$$

Next, note that L is decomposed as

$$L = \bigoplus_{\substack{a \in \mathbb{Z} \\ b, n \in \mathbb{Z}_{\geq 0}}} \bigoplus_{\lambda \in \Lambda^j} (L_{a,b,n} \cap L_\lambda);$$

here, recall that L_λ is the weight space of weight λ , and $L_{a,b,n}$ is defined as at the beginning of this subsection. Since $D := \bigoplus_{c \in \mathbb{Z}_{\geq 0}} \mathbb{Q}(p, q) \tilde{f}_{2'}^c(v) = \bigoplus_{c \in \mathbb{Z}_{\geq 0}} (L_{a_1+c, b_1, 0} \cap L_{a_1\delta_1+a_2\delta_2-c\gamma_2})$, one can take the complementary subspace C to D with respect to the decomposition above of L . We define $\tilde{f}_{2'}$ and $\tilde{e}_{2'}$ to be zero on C .

Theorem 6.1.3. *Let $a_1 \in \mathbb{Z}$, $a_2, b_1, b_2 \in \mathbb{Z}_{\geq 0}$, $b_2 \leq a_2$, and let $v \in L(a_1, a_2; b_1, b_2)$ be a highest weight vector. Set*

$$\begin{aligned} \mathcal{L}(a_1, a_2; b_1, b_2) &:= \sum_{\substack{l \in \mathbb{Z}_{\geq 0} \\ i_1, \dots, i_l \in \{1, 2, 2'\}}} \mathbf{A}_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(v) = \sum_{\substack{l, c \in \mathbb{Z}_{\geq 0} \\ i_1, \dots, i_l \in \{1, 2\}}} \mathbf{A}_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \tilde{f}_{2'}^c(v), \\ \mathcal{B}(a_1, a_2; b_1, b_2) &:= \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(v) + q\mathcal{L}(a_1, a_2; b_1, b_2) \mid l \in \mathbb{Z}_{\geq 0}, i_1, \dots, i_l \in \{1, 2, 2'\} \setminus \{0\}\} \\ &= \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \tilde{f}_{2'}^c(v) + q\mathcal{L}(a_1, a_2; b_1, b_2) \mid l, c \in \mathbb{Z}_{\geq 0}, i_1, \dots, i_l \in \{1, 2\} \setminus \{0\}\}. \end{aligned}$$

Then, $(\mathcal{L}(a_1, a_2; b_1, b_2), \mathcal{B}(a_1, a_2; b_1, b_2))$ is a quasi- j -crystal basis of $L(a_1, a_2; b_1, b_2)$.

We will reformulate this theorem for a general r as Theorem 7.2.1, which is proved in Section 9. The next subsection is devoted to the preparation for the proof.

6.2. Preparation for the proof of Theorem 7.2.1. Let $M \in \mathcal{O}_{\text{int}}^j$. For the computation in this subsection, it is important to obtain the commutation relations among $f_{2,1}$, $f_{2,2}$, and $f_{2,3}$.

Lemma 6.2.1. *For each $m \in M_{a,b,n}$, we have*

$$\begin{aligned} f_{2,1}f_{2,2}m &= \frac{\{a-2b-2\}}{\{a-2b\}} f_{2,2}f_{2,1}m, \\ f_{2,3}f_{2,1}m &= \frac{[b]}{[b+2]} f_{2,1}f_{2,3}m, \\ f_{2,3}f_{2,2}m &= \frac{\{a-b-1\}}{\{a-b+1\}} f_{2,2}f_{2,3}m. \end{aligned}$$

Proof. This is an easy consequence of Proposition 6.1.2. □

Recall that V_2 denotes the vector representation of $\mathbf{U} = U_q(\mathfrak{sl}_5)$. Let us consider $M \otimes V_2$, for which we know all the h_1 -eigenvectors and eigenvalues by the proof of Proposition 5.2.1.

For $m \in M_{a,b,0}$, we define $m[\overline{0}]$, $m[\overline{1}]$, $m[\overline{-1}]$, $m[\overline{2}]$, $m[\overline{-2}] \in M \otimes V_2$ by

$$\begin{aligned} m[\overline{0}] &:= m \otimes u_0, \\ m[\overline{1}] &:= m \otimes u_1 - \frac{q^{-b+1}(q-q^{-1})}{\{a-b-1\}} f_1 m \otimes u_0 - pq^{a-2b} m \otimes u_{-1}, \\ m[\overline{-1}] &:= f_1 m \otimes u_0 - q^b [b] m \otimes u_{-1} - pq^{a-b-2} [b] m \otimes u_1, \\ m[\overline{2}] &:= m \otimes u_2, \\ m[\overline{-2}] &:= m \otimes u_{-2}. \end{aligned} \tag{10}$$

Proposition 6.2.2. *We have*

$$\begin{aligned} m[\overline{0}] &\in (M \otimes V_2)_{a+2, b+1, 0}, \quad m[\overline{1}] \in (M \otimes V_2)_{a-1, b, 0}, \quad m[\overline{2}] \in (M \otimes V_2)_{a, b, 0}, \\ m[\overline{-1}] &\in (M \otimes V_2)_{a-1, b-1, 0}, \quad m[\overline{-2}] \in (M \otimes V_2)_{a, b, 0}. \end{aligned}$$

Proof. The assertions for $m\boxed{0}$, $m\boxed{1}$, and $m\boxed{-1}$ follow from the proof of Proposition 5.2.1. The rest is by easy calculations. \square

Remark 6.2.3. If $m \neq 0$, then $m\boxed{0}, m\boxed{1}, m\boxed{2}, m\boxed{-2} \neq 0$. Also, $m\boxed{-1} = 0$ if and only if $f_1 m = 0$, since $f_1 m = 0$ is equivalent to $b = 0$.

Let us compute $f_{2,2}^{(c)}(m\boxed{i})$ for $c \in \mathbb{Z}_{\geq 0}$, $i = 0, \pm 1$, explicitly; we omit the cases $i = \pm 2$ since they are easy. We write only the results; the proof for $c = 1$ is easy, and by induction on c , the proof proceeds straightforwardly.

Lemma 6.2.4. *Let $m \in M_{a,b,0}$. Then,*

$$f_{2,2}^{(c)}(m\boxed{0}) = f_{2,2}^{(c)} m\boxed{0}.$$

Lemma 6.2.5. *Let $m \in M_{a,b,0}$. Then,*

$$\begin{aligned} f_{2,2}^{(c)}(m\boxed{1}) &= f_{2,2}^{(c)} m\boxed{1} + q^{a_2-c+1} f_{2,2}^{(c-1)} m\boxed{2} - pq^{a-2b} f_{2,2}^{(c-1)} m\boxed{-2} \\ &\quad - \frac{q^{-b+1}(q-q^{-1})}{\{a-b-1\}} f_{2,3} f_1 f_{2,2}^{(c-1)} m\boxed{0} \\ &\quad + \frac{q^{-b+1}(q-q^{-1})}{[b+1]} \frac{\{a-2b-1\}}{\{a-2b+c-3\}\{a-2b+c-2\}} f_{2,2}^{(c-1)} f_{2,1} m\boxed{-1}. \end{aligned}$$

Lemma 6.2.6. *Let $m \in M_{a,b,0}$. Then,*

$$f_{2,2}^{(c)}(m\boxed{-1}) = \frac{\{a-2b-1\}}{\{a-2b+c-1\}} f_{2,2}^{(c)} m\boxed{-1}.$$

Now, we turn to the case that M is irreducible. Fix $(a_1, a_2; b_1, b_2)$ and assume that Theorem 6.1.3 holds for this $(a_1, a_2; b_1, b_2)$. Let us investigate how $L := L(a_1, a_2; b_1, b_2) \otimes V_2$ decomposes into irreducible submodules. In the following, $v \in L(a_1, a_2; b_1, b_2)$ denotes a highest weight vector. Set

$$v_0 := v\boxed{0}, \quad v_1 := v\boxed{1}, \quad v_{-1} := v\boxed{-1}.$$

Recall that v_0 and v_1 are always nonzero, while $v_{-1} \neq 0$ if and only if $b_1 > 0$.

Proposition 6.2.7. *The following hold:*

- (1) $\mathbf{U}_2^j(v_0) \simeq L(a_1 + 2, a_2; b_1 + 1, b_2)$.
- (2) $\mathbf{U}_2^j(v_1) \simeq L(a_1 - 1, a_2 + 1; b_1, b_2)$.
- (3) $\mathbf{U}_2^j(v_{-1}) \simeq L(a_1 - 1, a_2 + 1; b_1 - 1, b_2 + 1)$ if $b_1 > 0$.

Proof. See Appendix B.3 \square

Let $a \in \mathbb{Z}$, and consider the one-dimensional \mathbf{U}_2^j -module $\mathbb{Q}(p, q)v_a$ defined by $k_1 v_a = q^a v_a$, $k_2 v_a = v_a$, $x_i v_a = 0$ for $i \in \{1, 2\}$ and $x \in \{f, h, e\}$; clearly, it is isomorphic to $L(a, 0; 0, 0)$. If we set $a = a_1 + a_2 - 2b_1 - 2b_2$, then we see that

$$\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix} + (b_1 + b_2) \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + (a_2 - b_2) \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

This equality and Proposition 6.2.7 show that $L(a_1, a_2; b_1, b_2)$ is realized as a submodule of $\mathbb{Q}(p, q)v_a \otimes V_2^{\otimes(a_2+b_1+b_2)}$.

Proposition 6.2.8. *Assume that $\tilde{f}_{2'}^c(v) \neq 0$ if and only if $0 \leq c \leq a_2 - b_2$. Then we have*

$$\begin{aligned} \tilde{f}_{2'}^c(v_0) &\neq 0 \text{ if and only if } 0 \leq c \leq a_2 - b_2, \\ \tilde{f}_{2'}^c(v_1) &\neq 0 \text{ if and only if } 0 \leq c \leq (a_2 + 1) - b_2, \\ \tilde{f}_{2'}^c(v_{-1}) &\neq 0 \text{ if and only if } 0 \leq c \leq (a_2 + 1) - (b_2 + 1). \end{aligned}$$

Moreover, $\tilde{f}_{2'}^c(v_j) \in \mathcal{L} := \mathcal{L}(a_1, a_2; b_1, b_2) \otimes \mathcal{L}_2$, and $\tilde{f}_{2'}^c(v_j) + q\mathcal{L} \in \mathcal{B} := \mathcal{B}(a_1, a_2; b_1, b_2) \otimes \mathcal{B}_2$ for $j = 0, \pm 1$.

Proof. The assertions for v_0 and v_{-1} are clear from Lemma 6.2.4 and Lemma 6.2.6:

$$\begin{aligned} \tilde{f}_{2'}^c(v_0) &= q^{-cb_2} f_{2,2}^{(c)}(v[0]) = q^{-cb_2} f_{2,2}^{(c)}(v) \otimes u_0 = \tilde{f}_{2'}^c(v) \otimes u_0, \\ \tilde{f}_{2'}^c(v_{-1}) &= q^{-c(b_2+1)} f_{2,2}^{(c)}(v[-1]) \\ &= q^{-c(b_2+1)} \prod_{n=1}^c \frac{\{a_1 - 2b_1 - 2 + n\}}{\{a_1 - 2b_1 - 1 + n\}} f_{2,2}^{(c)}(v)[-1] \\ &= \frac{q^{-c}\{a_1 - 2b_1 - 1\}}{\{a_1 - 2b_1 + c - 1\}} \tilde{f}_{2'}^c(v)[-1] \in (1 + p\mathbf{A}_0) \tilde{f}_{2'}^c(v)[-1]. \end{aligned}$$

For $j = 1$, by Lemma 6.2.5, we have

$$\begin{aligned} \tilde{f}_{2'}^c(v_1) &= q^{-cb_2} f_{2,2}^{(c)}(v[1]) \\ &= q^{-cb_2} \left(f_{2,2}^{(c)}(v)[1] + q^{a_2-c+1} f_{2,2}^{(c-1)}(v)[2] - pq^{a_1-2b_1} f_{2,2}^{(c-1)}(v)[-2] \right. \\ &\quad - \frac{q^{-b_1+1}(q-q^{-1})}{\{a_1-b_1-1\}} f_{2,3} f_1 f_{2,2}^{(c-1)}(v)[0] \\ &\quad \left. + \frac{q^{-b_1+1}(q-q^{-1})}{[b_1+1]\{a_1-2b_1-2\}} f_{2,1} f_{2,2}^{(c-1)}(v)[-1] \right) \\ &= \tilde{f}_{2'}^c(v)[1] + q^{a_2-b_2-c+1} \tilde{f}_{2'}^{c-1}(v)[2] - pq^{a_1-2b_1-b_2} \tilde{f}_{2'}^{c-1}(v)[-2] \\ &\quad - \frac{q^{-b_1-b_2+1}(q-q^{-1})}{\{a_1-b_1-1\}} f_{2,3} f_1 \tilde{f}_{2'}^{c-1}(v)[0] \\ &\quad + \frac{q^{-b_1-b_2+1}(q-q^{-1})}{[b_1+1]\{a_1-2b_1-2\}} f_{2,1} \tilde{f}_{2'}^{c-1}(v)[0]; \end{aligned}$$

note that $f_1 \tilde{f}_{2'}^{c-1}(v) = \tilde{f}_1 \tilde{f}_{2'}^{c-1}(v) \in \mathcal{L}$. Since \mathcal{L} is a quasi- \mathcal{J} -crystal lattice, it follows that $f_2(\mathcal{L}) \subset q^{-N}\mathcal{L}$ for sufficiently large $N \geq 0$. Also, by the complete reducibility of $\mathbf{U}_1^{\mathcal{J}}$ -modules in $\mathcal{O}_{\text{int}}^{\mathcal{J}}$, \mathcal{L} is decomposed as $\mathcal{L} = \bigoplus_{a,b,n} \mathcal{L}_{a,b,n}$, where $\mathcal{L}_{a,b,n} := L_{a,b,n} \cap \mathcal{L}$. Suppose that $m \in \mathcal{L}_{a,b,n}$, and consider $f_2 m = f_{2,1}(m) + f_{2,2}(m) + f_{2,3}(m)$. Since $f_{2,i}(m) \in L_{a_i, b_i, n_i}$ for some a_i, b_i, n_i , it follows that $f_{2,i}(m) \in q^{-N}\mathcal{L}$ for sufficiently large $N \geq 0$. In particular, $\frac{q^{-b_1-b_2+1}(q-q^{-1})}{\{a_1-b_1-1\}} f_{2,3} f_1 \tilde{f}_{2'}^{c-1}(v)$ and $\frac{q^{-b_1-b_2+1}(q-q^{-1})}{[b_1+1]\{a_1-2b_1-2\}} f_{2,1} \tilde{f}_{2'}^{c-1}(v)$ belong to $p\mathcal{L}$. Therefore, we deduce that $\tilde{f}_{2'}^c(v[1]) \in \mathcal{L}$, and

$$\tilde{f}_{2'}^c(v_1) + q\mathcal{L} = \begin{cases} \tilde{f}_{2'}^c(v) \otimes u_1 + q\mathcal{L} & \text{if } c \leq a_2 - b_2, \\ \tilde{f}_{2'}^{a_2-b_2}(v) \otimes u_2 + q\mathcal{L} & \text{if } c = a_2 - b_2 + 1. \end{cases}$$

This proves the proposition. \square

Corollary 6.2.9. *We have $\tilde{f}_{2'}^c(v) \neq 0$ if and only if $0 \leq c \leq a_2 - b_2$*

Proof. It is clear that $\tilde{f}_{2'}^c(v_a) \neq 0$ if and only if $c = 0$. Hence, the assertion follows inductively by Proposition 6.2.8. \square

The submodule generated by v_0, v_{-1} , and v_1 may not be the whole of L . We find the other highest weight vectors as follows.

Let $\lambda \in \Lambda^{\mathcal{J}}$ be the highest weight of $L(a_1, a_2; b_1, b_2)$, i.e., $\lambda = a_1\delta_1 + a_2\delta_2$. Note that every weight of L is less than or equal to $\lambda + 2\delta_1$ with respect to the partial order defined by equation (6) in Section 2.3.

Lemma 6.2.10. *The subspace $L_{\lambda+2\delta_1-\gamma_1-\gamma_2} \cap \text{Ker}(e_1) \cap \text{Ker}(e_2) \cap \text{Ker}(h_1 - [b_1]\{a_1 - b_1 - 1\})$ is at most two-dimensional; it is spanned by*

$$\begin{aligned} m'_1 &:= f_{2,2}(v)[\underline{1}] - q^{-1}yv \otimes u_2 + \frac{q^{-b_1+1}(q - q^{-1})y}{\{a_1 - b_1\}z} (f_{2,3}f_1)(v)[\underline{0}] + pq^{a_1+a_2-2b_1+1}yv \otimes u_{-2}, \\ m'_2 &:= f_{2,1}(v)[\underline{-1}] + pq^{a_1-b_1-3}[b_1 + 1]xv \otimes u_2 - \frac{x}{z}(f_{2,3}f_1)(v)[\underline{0}] + q^{a_2+b_1+1}[b_1 + 1]xv \otimes u_{-2}, \end{aligned}$$

where we set

$$\begin{aligned} x &:= \frac{[b_2]\{a_1 + a_2 - 2b_1 - b_2 - 1\}}{\{a_1 - 2b_1 - 1\}}, \quad y := \frac{[a_2 - b_2]\{a_1 - 2b_1 - b_2 - 1\}}{\{a_1 - 2b_1 - 1\}}, \\ z &:= \frac{[b_1 + b_2 + 1]\{a_1 + a_2 - b_1 - b_2\}}{[b_1 + 1]\{a_1 - b_1\}}. \end{aligned}$$

Proof. See Appendix B.4. □

We would like to determine all the \mathbf{U}_2^j -highest weight vectors (or equivalently, h_2 -eigenvectors) in $L_{\lambda+2\delta_1-\gamma_1-\gamma_2} \cap \text{Ker}(e_1) \cap \text{Ker}(e_2) \cap \text{Ker}(h_1 - [b_1]\{a_1 - b_1 - 1\})$; this is completed in Proposition 6.2.11. If one of m'_1 and m'_2 is equal to zero, then we are done. Hence we assume that $m'_1 \neq 0$ and $m'_2 \neq 0$. Note that $m'_1 \neq 0$ if and only if $a_2 - b_2 \neq 0$, and that $m'_2 \neq 0$ if and only if $b_2 \neq 0$. For notational simplicity, we write $u \stackrel{q}{\sim} u'$ to indicate that $u - u' \in q\mathcal{L}'$ for $u, u' \in \mathcal{L}'$, where \mathcal{L}' is either $\mathcal{L}(a_1, a_2; b_1, b_2)$ or \mathcal{L} , and write $u \stackrel{q}{=} u'$ to indicate that $u' \in (1 + q\mathbf{A}_0)u$. Also, we write $u \stackrel{p}{\sim} u'$ to indicate that $q^N(u - u') \in p\mathcal{L}'$ for a sufficiently large $N \geq 0$. By setting

$$m_1 := -\frac{q}{y}m'_1, \quad m_2 := m'_2,$$

we see that $m_1 \stackrel{q}{=} v \otimes u_2$ and $m_2 \stackrel{q}{=} f_1 f_{2,1}(v) \otimes u_0$.

Proposition 6.2.11. *Assume that $a_2 - b_2 > 0$ and $b_2 > 0$.*

- (1) *There exists $v_2 \in \mathbf{A}_0 m_1 \oplus \mathbf{A}_0 m_2$ such that $v_2 - m_1 \in q(\mathbf{A}_0 m_1 \oplus \mathbf{A}_0 m_2)$ and $h_2 v_2 = [b_1 + b_2]\{a_1 + (a_2 - 1) - (b_1 + b_2) - 1\}v_2$. In particular, we have $\mathbf{U}_2^j v_2 \simeq L(a_1, a_2 - 1; b_1, b_2)$.*
- (2) *There exists $v_{-2} \in \mathbf{A}_0 m_1 \oplus \mathbf{A}_0 m_2$ such that $v_{-2} - m_2 \in q(\mathbf{A}_0 m_1 \oplus \mathbf{A}_0 m_2)$ and $h_2 v_{-2} = [b_1 + (b_2 - 1)]\{a_1 + (a_2 - 1) - (b_1 + (b_2 - 1)) - 1\}v_{-2}$. In particular, we have $\mathbf{U}_2^j v_{-2} \simeq L(a_1, a_2 - 1; b_1, b_2 - 1)$.*

Proof. See Appendix B.5. □

As the last step of the preparation for the proof of Theorem 7.2.1, we compute $\tilde{f}_{2'}^c(v_{\pm 2}) + q\mathcal{L}$. Since $m_1 \stackrel{p}{\sim} v \otimes u_2 - \frac{q}{y}f_{2,2}(v)[\underline{1}]$, we have $q^{-cb_2}f_{2,2}^{(c)}m_1 \stackrel{p}{\sim} q^{-cb_2}f_{2,2}^{(c)}(v \otimes u_2 - \frac{q}{y}f_{2,2}(v)[\underline{1}])$, and

$$\begin{aligned} q^{-cb_2}f_{2,2}^{(c)}(v \otimes u_2) &= \tilde{f}_{2'}^c(v) \otimes u_2, \\ q^{-cb_2}f_{2,2}^{(c)}(-\frac{q}{y}f_{2,2}(v)[\underline{1}]) &\stackrel{q}{\sim} -q^{a_2-b_2-cb_2}f_{2,2}^{(c)}(\tilde{f}_{2'}(v)[\underline{1}]) \\ &\stackrel{q}{\sim} -q^{a_2-b_2-c}\tilde{f}_{2'}^{c+1}(v)[\underline{1}] - q^{2(a_2-b_2-c)}\tilde{f}_{2'}^c(v)[\underline{2}]. \end{aligned}$$

Therefore, we deduce that

$$(11) \quad \tilde{f}_{2'}^c(m_1) \stackrel{q}{\sim} \begin{cases} \tilde{f}_{2'}^c(v) \otimes u_2 & \text{if } 0 \leq c \leq (a_2 - 1) - b_2, \\ 0 & \text{if } c = a_2 - b_2. \end{cases}$$

Similarly, we have $m_2 \stackrel{p}{\sim} f_{2,1}(v) \boxed{-1} - \frac{x}{z}(f_{2,3}f_1)v \boxed{0} + q^{a_2+b_1+1}[b_1+1]xv \boxed{-2}$, and

$$\begin{aligned} & q^{-c(b_2-1)} f_{2,2}^{(c)}(f_{2,1}(v) \boxed{-1} - \frac{x}{z}(f_{2,3}f_1)v \boxed{0} + q^{a_2+b_1+1}[b_1+1]xv \boxed{-2}) \\ &= q^{-c(b_2-1)} \left(\prod_{i=1}^c \frac{\{a_1 - 2b_1 - 3 + i\}}{\{a_1 - 2b_1 - 2 + i\}} f_{2,2}^{(c)} f_{2,1}(v) \boxed{-1} \right. \\ &\quad - \frac{x}{z} \prod_{i=1}^c \frac{\{a_1 - b_1 + i\}}{\{a_1 - b_1 + i - 1\}} (f_{2,3}f_1) f_{2,2}^{(c)}(v) \boxed{0} \\ &\quad \left. + q^{a_2+b_1+1-c}[b_1+1]x f_{2,2}^{(c)}(v) \boxed{-2} \right) \\ &= q^{-c(b_2-1)} \left(\frac{q^{cb_2}\{a_1 - 2b_1 + c - 1\}}{\{a_1 - 2b_1 - 1\}} f_{2,1} \tilde{f}_{2'}^c(v) \boxed{-1} \right. \\ &\quad - \frac{x}{z} \frac{q^{cb_2}\{a_1 - b_1 + c\}}{\{a_1 - b_1\}} (f_{2,3}f_1) \tilde{f}_{2'}^c(v) \boxed{0} \\ &\quad \left. + q^{a_2+b_1+1-c+cb_2}[b_1+1]x \tilde{f}_{2'}^c(v) \boxed{-2} \right). \end{aligned}$$

Therefore, we obtain

$$(12) \quad \tilde{f}_{2'}^c(m_2) \stackrel{q}{\sim} f_1 f_{2,1} \tilde{f}_{2'}^c(v) \otimes u_0 \quad \text{if } 0 \leq c \leq (a_2 - 1) - (b_2 - 1).$$

Proposition 6.2.12. *Assume that $a_2 - b_2 > 0$ and $b_2 > 0$. For $c \in \mathbb{Z}_{\geq 0}$, we have*

$$\begin{aligned} \tilde{f}_{2'}^c(v_2) + q\mathcal{L} &= \begin{cases} \tilde{f}_{2'}^c(v) \otimes u_2 & \text{if } 0 \leq c \leq (a_2 - 1) - b_2, \\ 0 & \text{if } c > (a_2 - 1) - b_2, \end{cases} \\ \tilde{f}_{2'}^c(v_{-2}) + q\mathcal{L} &= \begin{cases} \tilde{f}_1 \tilde{f}_2 \tilde{f}_{2'}^c(v) \otimes u_0 & \text{if } 0 \leq c \leq (a_2 - 1) - (b_2 - 1), \\ 0 & \text{if } c > (a_2 - 1) - (b_2 - 1). \end{cases} \end{aligned}$$

Proof. We prove the assertion for v_2 . By Corollary 6.2.9, we have $\tilde{f}_{2'}^c(v_2) = 0$ for $c > (a_2 - 1) - b_2$. Also, by equality (11) and (12), we have

$$\tilde{f}_{2'}^c(v_2) + q\mathcal{L} = \tilde{f}_{2'}^c(m_1) + q\mathcal{L} = \tilde{f}_{2'}^c(v_2) + q\mathcal{L}$$

for $0 \leq c \leq (a_2 - 1) - b_2$, as desired. The proof of the assertion for v_{-2} is similar. \square

7. CRYSTAL BASIS THEORY FOR \mathbf{U}^j

7.1. j -crystal bases.

Definition 7.1.1. Let A be an associative algebra over $\mathbb{Q}(p, q)$. We call $(f_1^A, f_2^A, k_1^A, k_2^A, e_1^A, e_2^A) \in A^6$ a \mathbf{U}_2^j -sextuple if there is an injective algebra homomorphism $\mathbf{U}_2^j \rightarrow A$ which sends $f_1, f_2, k_1, k_2, e_1, e_2$ to $f_1^A, f_2^A, k_1^A, k_2^A, e_1^A, e_2^A$, respectively.

Example 7.1.2. Consider $A = \mathbf{U}_r^j$. For each $i \in \{2, \dots, r\}$, the sextuple

$$(T_i(f_1), T_i(f_2), T_i(k_1), T_1(k_2), T_i(e_1), T_i(e_2)) \in (\mathbf{U}_r^j)^6$$

is a \mathbf{U}_2^j -sextuple, where $T_i := (\tau_{i-1}\tau_i) \cdots (\tau_3\tau_4)(\tau_2\tau_3)$ ($i > 2$), and $T_2 := \text{id}$. Note that $T_i(x_2) = x_i$ for $x \in \{e, k, f\}$, $T_i(k_1) = k_1 \cdots k_{i-1}$, $T_i(h_1) = h_{i-1}$, and $T_i(h_2) = h_i$.

Let us define linear maps $\tilde{f}_{i'}$ and $\tilde{e}_{i'}$, $i \in \mathbb{I} \setminus \{1\}$, on each $M \in \mathcal{O}_{\text{int}}^j$ as follows. Let $i \in \mathbb{I} \setminus \{1\}$, and L an irreducible \mathbf{U}_i^j -module with highest weight vector $v \in L$. Let $L^{(i)}$ denote the module L over the \mathbf{U}_2^j -sextuple $(T_i(f_1), T_i(f_2), T_i(k_1), T_1(k_2), T_i(e_1), T_i(e_2))$, that is, $L^{(i)}$ is the vector space L equipped with a \mathbf{U}_2^j -module structure via the homomorphism $T_i : \mathbf{U}_2^j \rightarrow \mathbf{U}_i^j$. Then, $\tilde{f}_{i'}$ (resp., $\tilde{e}_{i'}$) is defined to be $\tilde{f}_{2'}$ (resp., $\tilde{e}_{2'}$) on the irreducible component of the \mathbf{U}_2^j -module

$L^{(i)}$ containing v , and 0 on the other irreducible components. Note that for $c \in \mathbb{Z}_{\geq 0}$, we have $\tilde{f}_{i'}^c(v) = T_i \circ \tilde{f}_{2'} \circ T_i^{-1}(v)$.

Definition 7.1.3. Let M be a \mathbf{U}_r^j -module and \mathcal{L} an \mathbf{A}_0 -submodule of M . We say that \mathcal{L} is a j -crystal lattice of M if

- (L 1) \mathcal{L} is a quasi- j -crystal lattice of M ,
- (L 2) $\tilde{f}_{i'}(\mathcal{L}) \subset \mathcal{L}$ and $\tilde{e}_{i'}(\mathcal{L}) \subset \mathcal{L}$ for all $i \in \mathbb{I}^j \setminus \{1\}$.

If \mathcal{L} is a j -crystal lattice of M , then the Kashiwara operators $\tilde{f}_{i'}$ induce \mathbb{Q} -linear maps, denoted by the same symbols, on $\mathcal{L}/q\mathcal{L}$.

Definition 7.1.4. Let M be a \mathbf{U}_r^j -module, \mathcal{L} an \mathbf{A}_0 -submodule of M , and \mathcal{B} a subset of $\mathcal{L}/q\mathcal{L}$. We say that $(\mathcal{L}, \mathcal{B})$ is a j -crystal basis if

- (B 1) \mathcal{L} is a j -crystal lattice of M ,
- (B 2) $(\mathcal{L}, \mathcal{B})$ is a quasi- j -crystal basis of M ,
- (B 3) for each $b, b' \in \mathcal{B}$ and $i \in \mathbb{I}^j \setminus \{1\}$, one has $\tilde{f}_{i'}(b) = b'$ if and only if $b = \tilde{e}_{i'}(b')$.

In order to describe $\tilde{f}_{i'}$, we use a symmetry of quasi- j -crystal bases. Let \mathfrak{S}_{r-1} denote the $(r-1)$ -st symmetric group with simple reflections $\{s_i \mid i = 2, \dots, r\}$.

Lemma 7.1.5. Let $(\mathcal{L}, \mathcal{B})$ be a quasi- j -crystal basis of a \mathbf{U}_r^j -module M . Then \mathfrak{S}_{r-1} acts on \mathcal{B} as follows:

$$s_i(b) := \begin{cases} \tilde{f}_i^{(\beta_i, \text{wt}(b))}(b) & \text{if } (\beta_i, \text{wt}(b)) \geq 0, \\ \tilde{e}_i^{-(\beta_i, \text{wt}(b))}(b) & \text{if } (\beta_i, \text{wt}(b)) < 0. \end{cases}$$

Proof. Since the subalgebra of \mathbf{U}^j generated by $\{f_i, k_i, e_i \mid i \in \mathbb{I}^j \setminus \{1\}\}$ is isomorphic to $U_q(\mathfrak{sl}_r)$, the $(\mathcal{L}, \mathcal{B})$ is equipped with a $U_q(\mathfrak{sl}_r)$ -crystal structure by ignoring the actions of \tilde{f}_1 and \tilde{e}_1 . Hence, the assertion follows from the ordinary crystal basis theory for quantum groups. \square

For convenience, we introduce operators \tilde{f}_i^{\max} and \tilde{e}_i^{\max} , $i \in \mathbb{I}^j$, acting on \mathbf{U}^j -modules. Let $M \in \mathcal{O}_{\text{int}}^j$ and $m \in M$. By the definition of $\mathcal{O}_{\text{int}}^j$, there exists a unique integer N_i such that $\tilde{f}_i^{N_i}(m) \neq 0$ and $\tilde{f}_i^{N_i+1}(m) = 0$. Then, $\tilde{f}_i^{\max}(m)$ is defined to be $\tilde{f}_i^{N_i}(m)$. The \tilde{e}_i^{\max} is defined in a similar way.

Lemma 7.1.6. Let M be a \mathbf{U}_r^j -module and $m \in M$ a \mathbf{U}_i^j -highest weight vector. For each $c \in \mathbb{Z}_{\geq 0}$, we have $\tilde{f}_{i'}^c(m) = (\tilde{e}_{i-1}^{\max} \tilde{e}_i^{\max}) \cdots (\tilde{e}_2^{\max} \tilde{e}_3^{\max}) \tilde{f}_{2'}^c(\tilde{f}_3^{\max} \tilde{f}_2^{\max}) \cdots (\tilde{f}_i^{\max} \tilde{f}_{i-1}^{\max})(m)$. In particular, if M admits a j -crystal basis $(\mathcal{L}, \mathcal{B})$ and if we set $b := m + q\mathcal{L} \in \mathcal{B} \setminus \{0\}$, then $\tilde{f}_{i'}^c(b) = (s_{i-1}s_i) \cdots (s_2s_3) \tilde{f}_{2'}^c(s_3s_2) \cdots (s_i s_{i-1})(b)$.

Proof. We prove the assertion by induction on $i \geq 2$. The case $i = 2$ is trivial. Hence let $i \geq 3$, and assume that the assertion holds for $i-1$. Since m is a \mathbf{U}_i^j -highest weight vector, $(\tau_{i-1}\tau_i)^{-1}(m) = \tilde{f}_i^{\max} \tilde{f}_{i-1}^{\max}(m)$ is a \mathbf{U}_{i-1}^j -highest weight vector with highest weight $(a_1, \dots, a_{i-3}, a_{i-2} + a_{i-1}, a_i; b_1, \dots, b_{i-3}, b_{i-2} + b_{i-1}, b_i)$. Therefore, we have

$$\begin{aligned} q^{cb_i} \tilde{f}_{i'}^c(m) &= f_{i,2}^{(c)}(m) \\ &= T_i f_{2,2}^{(c)} T_i^{-1}(m) \\ &= (\tau_{i-1}\tau_i) T_{i-1} f_{2,2}^{(c)} T_{i-1}^{-1} (\tau_{i-1}\tau_i)^{-1}(m) \\ &= (\tau_{i-1}\tau_i) f_{i-1,2}^{(c)} (\tau_{i-1}\tau_i)^{-1}(m) \\ &= q^{cb_i} (\tau_{i-1}\tau_i) \tilde{f}_{(i-1)'}^c (\tau_{i-1}\tau_i)^{-1}(m). \end{aligned}$$

Let $\lambda \in \Lambda^j$ be the weight of m . Then, the weight of $\tilde{f}_{(i-1)'}^c (\tau_{i-1}\tau_i)^{-1}(m)$ is equal to $s_i s_{i-1}(\lambda) - c\gamma_{i-1}$. Since m is a \mathbf{U}_i^j -highest weight vector, every weight of the submodule $\mathbf{U}_i^j m$ is less than

or equal to λ . Noting this, we can show that $f_i \tilde{f}_{(i-1)'}^c (\tau_{i-1} \tau_i)^{-1}(m) = 0$. Indeed, if not, then $s_i s_{i-1}(\lambda) - c\gamma_{i-1} - \gamma_i$ is a weight of $\mathbf{U}_i^j m$, and hence so is $s_{i-1} s_i(s_i s_{i-1}(\lambda) - c\gamma_{i-1} - \gamma_i) = \lambda + \gamma_{i-1} - (c-1)\gamma_i \not\leq \lambda$, which is a contradiction. Similarly, $f_{i-1} \tau_i \tilde{f}_{(i-1)'}^c (\tau_{i-1} \tau_i)^{-1}(m) = 0$. Hence, we compute as follows:

$$\begin{aligned} & (\tau_{i-1} \tau_i) \tilde{f}_{(i-1)'}^c (\tau_{i-1} \tau_i)^{-1}(m) \\ &= (\tau_{i-1} \tau_i) (\tilde{e}_{i-2}^{\max} \tilde{e}_{i-1}^{\max}) \cdots (\tilde{e}_2^{\max} \tilde{e}_3^{\max}) \tilde{f}_{2'}^c (\tilde{f}_3^{\max} \tilde{f}_2^{\max}) \cdots (\tilde{f}_{i-1}^{\max} \tilde{f}_{i-2}^{\max}) (\tau_{i-1} \tau_i)^{-1}(m) \\ &= (\tilde{e}_{i-1}^{\max} \tilde{e}_i^{\max}) (\tilde{e}_{i-2}^{\max} \tilde{e}_{i-1}^{\max}) \cdots (\tilde{e}_2^{\max} \tilde{e}_3^{\max}) \tilde{f}_{2'}^c (\tilde{f}_3^{\max} \tilde{f}_2^{\max}) \cdots (\tilde{f}_{i-1}^{\max} \tilde{f}_{i-2}^{\max}) (\tilde{f}_i^{\max} \tilde{f}_{i-1}^{\max})(m). \end{aligned}$$

This proves the lemma. \square

Set $\overline{\mathbb{I}^j} := \mathbb{I}^j \sqcup \{i' \mid 2 \leq i \leq r\}$.

Definition 7.1.7. Let M be a \mathbf{U}_r^j -module and $(\mathcal{L}, \mathcal{B})$ a j -crystal basis of M . The j -crystal graph associated with $(\mathcal{L}, \mathcal{B})$ is a colored directed graph with vertex set \mathcal{B} and edges $b \xrightarrow{i} b'$, where $b, b' \in \mathcal{B}$, $i \in \overline{\mathbb{I}^j}$ are such that $\tilde{f}_i b = b'$.

7.2. Existence and uniqueness theorem. We are ready to state one of our main results in this paper; this gives the existence and uniqueness of a j -crystal basis of $L(\mathbf{a}; \mathbf{b})$.

Theorem 7.2.1. Consider the irreducible highest weight module $L(\mathbf{a}; \mathbf{b})$ and take a highest weight vector $v \in L(\mathbf{a}; \mathbf{b})$. Set

$$\begin{aligned} \mathcal{L}(\mathbf{a}; \mathbf{b}) &:= \sum_{\substack{l \in \mathbb{Z}_{\geq 0} \\ i_1, \dots, i_l \in \overline{\mathbb{I}^j}}} \mathbf{A}_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(v) = \sum_{\substack{l, c \in \mathbb{Z}_{\geq 0} \\ i_1, \dots, i_l \in \overline{\mathbb{I}^j} \setminus \{r'\}}} \mathbf{A}_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \tilde{f}_{r'}^c(v), \\ \mathcal{B}(\mathbf{a}, \mathbf{b}) &:= \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(v) + q\mathcal{L}(\mathbf{a}; \mathbf{b}) \mid l \in \mathbb{Z}_{\geq 0}, i_1, \dots, i_l \in \overline{\mathbb{I}^j}\} \setminus \{0\} \\ &= \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \tilde{f}_{r'}^c(v) + q\mathcal{L}(\mathbf{a}; \mathbf{b}) \mid l, c \in \mathbb{Z}_{\geq 0}, i_1, \dots, i_l \in \overline{\mathbb{I}^j} \setminus \{r'\}\} \setminus \{0\}. \end{aligned}$$

Then the following hold:

- (1) $(\mathcal{L}(\mathbf{a}; \mathbf{b}), \mathcal{B}(\mathbf{a}; \mathbf{b}))$ is a j -crystal basis of $L(\mathbf{a}; \mathbf{b})$.
- (2) Let M be a \mathbf{U}^j -module having a j -crystal basis $(\mathcal{L}, \mathcal{B})$. Suppose that $M \simeq \bigoplus_{t \in T} L(\mathbf{a}_t; \mathbf{b}_t)$, where T is an index set. Then, there exists an isomorphism $\phi : M \rightarrow \bigoplus_{t \in T} L(\mathbf{a}_t; \mathbf{b}_t)$ of \mathbf{U}^j -modules which induces an isomorphism $(\mathcal{L}, \mathcal{B}) \rightarrow (\bigoplus_{t \in T} \mathcal{L}(\mathbf{a}_t; \mathbf{b}_t), \bigsqcup_{t \in T} \mathcal{B}(\mathbf{a}_t; \mathbf{b}_t))$ of j -crystal bases.

We prove this theorem in Section 9 after introducing some combinatorial tools in the next section.

8. EXPLICIT DESCRIPTION OF j -CRYSTAL BASES

8.1. Double partitions and double Young tableaux.

Definition 8.1.1. Let N be a nonnegative integer. A partition $\alpha = (\alpha_1, \dots, \alpha_l)$ of N is a nonincreasing sequence of nonnegative integers $\alpha_1 \geq \dots \geq \alpha_l \geq 0$ such that $\sum_{i=1}^l \alpha_i = N$. We call $|\alpha| := N$ the size of α , and $\ell(\alpha) := l$ the length of α .

Definition 8.1.2. Let α be a partition of N . The Young diagram $D(\alpha)$ associated with α is the set $\{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq i \leq l, 1 \leq j \leq \alpha_i\}$. Note that the Young diagram $D(0, 0, \dots, 0)$ is the empty set.

We often identify a partition α with its Young diagram $D(\alpha)$.

Definition 8.1.3. Let α be a partition of N . A Young tableau T of shape α is a map from $D(\alpha)$ to a totally ordered set. A Young tableau T is said to be semistandard if it satisfies $T(i, j) \leq T(i, l)$ and $T(i, j) < T(k, j)$ for all $(i, j), (i, l), (k, j) \in D(\alpha)$ such that $j < l$, $i < k$. A semistandard Young tableau T is said to be standard if it satisfies $T(i, j) < T(i, l)$ for all $(i, j), (i, l) \in D(\alpha)$ such that $j < l$, and if $T(D(\alpha)) = \{1, 2, \dots, |\alpha|\}$.

Definition 8.1.4. Let N be a nonnegative integer. A double partition $(\alpha; \beta)$ of N is an ordered pair of partitions such that $|\alpha| + |\beta| = N$. We call N the size of $(\alpha; \beta)$, and $(\ell(\alpha); \ell(\beta))$ the length of $(\alpha; \beta)$; we denote the size of $(\alpha; \beta)$ by $|\alpha; \beta|$.

Note that we distinguish between $(\alpha; \beta)$ and $(\beta; \alpha)$; in particular, for a partition α , the pairs $(\alpha; \emptyset)$ and $(\emptyset; \alpha)$ are distinct double partitions.

Let $L(\mathbf{a}; \mathbf{b})$ be an irreducible highest weight \mathbf{U}^j -module; namely, $\mathbf{a} = (a_1, \dots, a_r)$, $\mathbf{b} = (b_1, \dots, b_r)$, with $a_1 \in \mathbb{Z}$, $a_2, \dots, a_r, b_1, \dots, b_r \in \mathbb{Z}_{\geq 0}$, and $0 \leq b_i \leq a_i$ for $i = 2, \dots, r$. Set

$$\begin{aligned} \alpha &:= \left(\sum_{i=1}^r b_i, \sum_{i=2}^r b_i, \dots, b_r, 0 \right) + a_1^+ \rho_{r+1}, \\ \beta &:= \left(\sum_{i=2}^r (a_i - b_i), \sum_{i=3}^r (a_i - b_i), \dots, a_r - b_r, 0 \right) - a_1^- \rho_r, \end{aligned}$$

where $a_1^+ := \max\{a_1 - (2 \sum_{i=1}^r b_i - \sum_{i=2}^r a_i), 0\}$, $a_1^- := \min\{a_1 - (2 \sum_{i=1}^r b_i - \sum_{i=2}^r a_i), 0\}$, $\rho_n := (1, 1, \dots, 1)$ (n components), and the addition is defined componentwise. The assignment $(\mathbf{a}; \mathbf{b}) \mapsto (\alpha; \beta)$ gives a bijection from $\{(\mathbf{a}; \mathbf{b}) \mid a_1 \in \mathbb{Z}, b_1 \in \mathbb{Z}_{\geq 0}, 0 \leq b_i \leq a_i, i \geq 2\}$ to the set of double partitions of length $(r+1; r)$ containing at least one 0; the inverse map π is given by

$$(13) \quad a_1 = 2\alpha_1 - \alpha_2 - \beta_1, \quad a_i = \alpha_i - \alpha_{i+1} + \beta_{i-1} - \beta_i, \quad b_i = \alpha_i - \alpha_{i+1}.$$

We write $L(\alpha; \beta) = L(\pi(\mathbf{a}; \mathbf{b}))$. If we define $\pi(\alpha; \beta)$ by equation (13) for a double partition $(\alpha; \beta)$ of length $(r+1; r)$, then $\pi(\alpha; \beta) = \pi(\alpha'; \beta')$ if and only if $(\alpha'; \beta') = (\alpha + n\rho_{r+1}; \beta + n\rho_r)$ for some $n \in \mathbb{Z}$. We denote this condition by $(\alpha; \beta) \sim_\pi (\alpha'; \beta')$, and define $L(\alpha'; \beta')$ to be $L(\alpha; \beta)$, where $(\alpha; \beta)$ is the unique double partition containing at least one 0 such that $(\alpha; \beta) \sim_\pi (\alpha'; \beta')$. From these observations, we obtain the following.

Proposition 8.1.5. *The isomorphism classes of irreducible \mathbf{U}^j -modules in $\mathcal{O}_{\text{int}}^j$ are parametrized by the double partitions of length $(r+1; r)$ modulo \sim_π .*

Definition 8.1.6. Let $(\alpha; \beta)$ be a double partition of N . The double Young diagram $D(\alpha; \beta)$ associated with $(\alpha; \beta)$ is the ordered pair $(D(\alpha); D(\beta))$; we often identify a double partition with its double Young diagram.

Example 8.1.7. Let $r = 3$, $\mathbf{a} = (2, 2, 3)$, $\mathbf{b} = (2, 0, 1)$. Then the corresponding double partition is $(4, 2, 2, 1; 4, 2, 0)$, and the associated double Young diagram is

$$\left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} ; \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right).$$

Definition 8.1.8. For $s \in \{-r, -r+1, \dots, r\}$, a double partition $(\alpha; \beta)$ is said to be s -addable if $\alpha_{|s|+1} < \alpha_{|s|}$ when $s \leq 0$, and $\beta_s < \beta_{s-1}$ when $s > 0$. Here we understand that $\alpha_0 = \beta_0 = \infty$ by convention.

Example 8.1.9. A double partition $(4, 2, 2, 1; 4, 2, 0)$ is s -addable for $s = -3, -1, 0, 1, 2, 3$.

Definition 8.1.10. Let $(\alpha; \beta)$ be a double partition of N . A double Young tableau $(T_1; T_2)$ of shape $(\alpha; \beta)$ is an ordered pair of a Young tableau T_1 of shape α and a Young tableau T_2 of shape β . A double Young tableau is said to be semistandard if T_1 and T_2 are both semistandard.

Definition 8.1.11. We denote by $\text{SST}_r(\alpha; \beta)$ the set of double Young tableaux $(T_1; T_2)$ of shape $(\alpha; \beta)$ such that $T_1(i, j) \in \{0, -1, \dots, -r\}$ and $T_2(i, j) \in \{1, \dots, r\}$. Here, we equip $\{0, -1, \dots, -r\}$ with a total order $0 \prec -1 \prec \dots \prec -r$.

Note that there exists a natural bijection $\text{SST}_r(\alpha; \beta) \rightarrow \text{SST}_r(\alpha + n\rho_{r+1}; \beta + n\rho_r)$ for all $n \in \mathbb{N}$.

8.2. \mathcal{J} -crystal structure on $\mathcal{B}_r^{\otimes N}$. Recall that $\mathcal{B}_r^{\otimes N} = \{-r, \dots, r\}^N$ is equipped with a quasi- \mathcal{J} -crystal structure described in Section 5.2. There, we showed that for $\mathbf{s} \in \mathcal{B}_r^{\otimes N}$ and $i \in \mathbb{I}$, the $\tilde{x}_i(\mathbf{s})$, $x \in \{e, f\}$, is of the form $\tilde{X}_{\pm(i-\frac{1}{2})}(\mathbf{s})$ or $\tilde{X}_{\pm(i-\frac{1}{2})}(\mathbf{s})$, $X \in \{E, F\}$. Now, for $i \in \mathbb{I} \setminus \{1\}$, we define maps $\tilde{f}_{i'}, \tilde{e}_{i'} : \mathcal{B}_r^{\otimes N} \rightarrow \mathcal{B}_r^{\otimes N} \sqcup \{0\}$ as follows. Let $\mathbf{s} = (s_1, \dots, s_N) \in \mathcal{B}_r^{\otimes N}$, and recall the definition of $\mathbf{s}_{i-\frac{1}{2}}$ from Section 5.2. Then, $\tilde{f}_{i'}(\mathbf{s})$ (resp., $\tilde{e}_{i'}(\mathbf{s})$) is defined to be the element obtained from \mathbf{s} by replacing the leftmost $i-1$ (resp., the rightmost i) in $\mathbf{s}_{i-\frac{1}{2}}$ with i (resp., $i-1$) if $i-1 \in \mathbf{s}_{i-\frac{1}{2}}$ (resp., $i \in \mathbf{s}_{i-\frac{1}{2}}$) and $\tilde{e}_j(\mathbf{s}) = 0$ for all $j = 1, \dots, i, 2', \dots, (i-1)'$; otherwise, $\tilde{f}_{i'}(\mathbf{s})$ (resp., $\tilde{e}_{i'}(\mathbf{s})$) is defined to be 0. Note that $\tilde{f}_{i'}(\mathbf{s})$ equals either $\tilde{F}_{i-\frac{1}{2}}(\mathbf{s})$ or 0, and that $\tilde{e}_{i'}(\mathbf{s})$ equals either $\tilde{E}_{i-\frac{1}{2}}(\mathbf{s})$ or 0.

Remark 8.2.1. In Section 9, we prove that $V_r^{\otimes N}$ has a \mathcal{J} -crystal basis $(\mathcal{L}_r^{\otimes N}, \mathcal{B}_r^{\otimes N})$, on which the Kashiwara operators \tilde{f}_i and \tilde{e}_i , $i \in \overline{\mathbb{I}}$, act as we described above.

Let $(\alpha; \beta)$ be a double partition of size N of length $(r+1, r)$. Consider the map $\text{SST}_r(\alpha; \beta) \rightarrow \mathcal{B}_r^N$ given by the assignment $(T_1; T_2) \mapsto (\text{EM}(T_1), \text{ME}(T_2))$, where $\text{ME}(T_2)$ means the Middle-Eastern reading of T_2 , and $\text{EM}(T_1)$ is obtained by reversing $\text{ME}(T_1)$. For example,

$$\left(\begin{array}{|c|c|c|c|} \hline 0 & 0 & -2 & -3 \\ \hline -1 & -1 & & \\ \hline -2 & -3 & & \\ \hline -4 & & & \\ \hline \end{array} ; \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 4 & & \\ \hline \end{array} \right) \mapsto (-4, -2, -3, -1, -1, 0, 0, -2, -3, 2, 2, 1, 1, 4, 2).$$

For $(T_1; T_2) \in \text{SST}_r(\alpha; \beta)$, we define $\tilde{f}_i(T_1; T_2)$ to be the unique (not necessarily semistandard) double Young tableau $(T'_1; T'_2)$ of shape $(\alpha; \beta)$ such that $(\text{EM}(T'_1), \text{ME}(T'_2)) = (\tilde{f}_i(\text{EM}(T_1), \text{ME}(T_2)))$ for $i \in \overline{\mathbb{I}}$. The double Young tableau $\tilde{e}_i(T_1; T_2)$ is defined similarly. By the first paragraph of this subsection, for each $i \in \overline{\mathbb{I}}$, the $\tilde{x}_i(T_1; T_2)$, $x \in \{e, f\}$, is of the form $(\tilde{X}_{-(i-\frac{1}{2})}(T_1; T_2))$ or $(T_1; \tilde{X}_{i-\frac{1}{2}}(T_2))$, $X \in \{E, F\}$. Therefore, $\text{SST}_r(\alpha; \beta) \sqcup \{0\}$ is stable under the operators \tilde{f}_i and \tilde{e}_i , $i \in \overline{\mathbb{I}}$. Clearly, there exists an isomorphism $\text{SST}_r(\alpha; \beta) \rightarrow \text{SST}_r(\alpha + n\rho_{r+1}; \beta + n\rho_r)$ of quasi- \mathcal{J} -crystal graphs which is compatible with $\tilde{f}_{i'}$ and $\tilde{e}_{i'}$, $i \in \mathbb{I} \setminus \{1\}$.

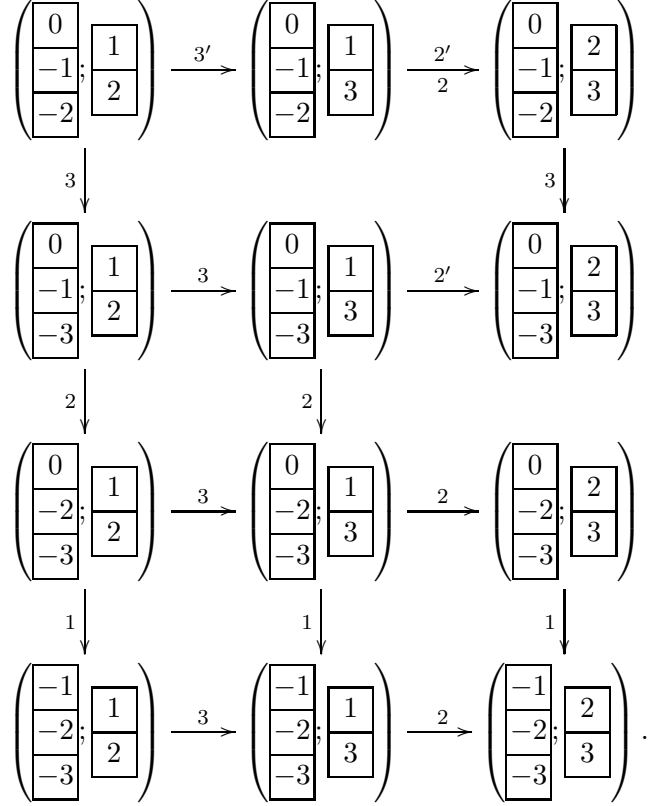
Now, we define $(T_\alpha; T_\beta) \in \text{SST}_r(\alpha; \beta)$ by $T_\alpha(i, j) = -(i-1)$, and $T_\beta(i, j) = i$. For example, when $\alpha = (4, 2, 2, 1)$ and $\beta = (4, 2, 0)$,

$$(T_\alpha; T_\beta) = \left(\begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline -1 & -1 & & \\ \hline -2 & -2 & & \\ \hline -3 & & & \\ \hline \end{array} ; \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & & \\ \hline \end{array} \right).$$

Proposition 8.2.2. *For each $(T_1; T_2) \in \text{SST}_r(\alpha; \beta)$, there exists a sequence $i_1, \dots, i_l \in \overline{\mathbb{I}}$ such that $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(T_1; T_2) = (T_\alpha; T_\beta)$.*

Proof. Let $(T_1; T_2) \in \text{SST}_r(\alpha; \beta)$. Suppose that $\tilde{e}_i(T_1; T_2) = 0$ for all $i \in \mathbb{I}$. By the tensor product rule, this implies that $\tilde{F}_{-(i-\frac{1}{2})}(T_1) = 0$ for all $i \in \mathbb{I}$, or equivalently, $T_1 = T_\alpha$. Set $d(T_2) := \sum_{i,j} (T_2(i, j) - T_\beta(i, j))$; This measures the distance between T_2 and T_β , that is, one has $d(T_2) \geq 0$, and the equality holds if and only if $T_2 = T_\beta$. Suppose that $d(T_2) > 0$. Then there exists a minimal $i_1 \in \mathbb{I} \setminus \{1\}$ such that $\tilde{E}_{i_1-\frac{1}{2}}(T_2) \neq 0$. For such i_1 , we have $\tilde{e}_{i_1'}(T_\alpha, T_2) = (T_\alpha, \tilde{E}_{i_1-\frac{1}{2}}(T_2))$ and $d(\tilde{E}_{i_1-\frac{1}{2}}(T_2)) = d(T_2) - 1$. Repeating this, we obtain a sequence (i_1, \dots, i_l) such that $\tilde{e}_{i_1'} \cdots \tilde{e}_{i_l'}(T_\alpha; T_2) = (T_\alpha; T_\beta)$. This proves the proposition. \square

Example 8.2.3. Let $r = 3$, $\alpha = (1, 1, 1, 0)$, $\beta = (1, 1, 0)$. Then, the j -crystal graph of $\text{SST}_3(\alpha; \beta)$ is as follows:



Note that the quasi- j -crystal graph of $\text{SST}_3(\alpha; \beta)$, which is obtained by removing the directed edges colored by $2', 3'$, is not connected.

9. PROOF OF THEOREM 7.2.1

Let $N \in \mathbb{N}$, and consider $V_r^{\otimes N}$. Following the proof of the existence of crystal bases in [Ka91], we prove the existence of j -crystal bases by decomposing $V_r^{\otimes N}$ into irreducible modules. We can write $V_r^{\otimes N} \simeq \bigoplus_{t \in T} L(\alpha_t; \beta_t)$, where T is an index set and $(\alpha_t; \beta_t)$ is a double partition of length $(r+1; r)$. In this section, we prove

$A_r(N)$: $V_r^{\otimes N}$ has a j -crystal basis $(\mathcal{L}_r^{\otimes N}, \mathcal{B}_r^{\otimes N})$, on which Kashiwara operators act as in the first paragraph of Section 8.2,

and that there exists an irreducible decomposition $V_r^{\otimes N} = \bigoplus_{t \in T} L_t$ of $V_r^{\otimes N}$ such that $L_t \simeq L(\alpha_t; \beta_t)$ for all $t \in T$, and satisfies the following conditions $B_r(N) - D_r(N)$.

$B_r(N)$: For $t \in T$, set $\mathcal{L}_t := \mathcal{L}_r^{\otimes N} \cap L_t$, $\mathcal{B}_t := \mathcal{B}_r^{\otimes N} \cap (\mathcal{L}_t / q\mathcal{L}_t)$. Then, we have $\mathcal{L}_r^{\otimes N} = \bigoplus_{t \in T} \mathcal{L}_t$, $\mathcal{B}_r^{\otimes N} = \bigsqcup_{t \in T} \mathcal{B}_t$. Moreover, for each $t \in T$, $(\mathcal{L}_t, \mathcal{B}_t)$ is a j -crystal basis of L_t , on which Kashiwara operators act as the restriction of those acting on $(\mathcal{L}_r^{\otimes N}, \mathcal{B}_r^{\otimes N})$.

$C_r(N)$: For each double partition $(\alpha; \beta)$ of N , there exists a unique $t \in T$ such that $(\alpha; \beta) = (\alpha_t; \beta_t)$, and the j -crystal graph of $(\mathcal{L}_t, \mathcal{B}_t)$ is connected with a single source $(\text{EM}(T_\alpha), \text{ME}(T_\beta))$.

$D_r(N)$: $|\alpha_t; \beta_t| = N$ for all $t \in T$.

If we are done, then Theorem 7.2.1 is proved as follows. Let $(\alpha; \beta)$ be a double partition of N of length $(r+1; r)$, and $v \in L(\alpha; \beta)$ a highest weight vector. By assertion $C_r(N)$, we may assume that $L(\alpha; \beta) = L_t$ for some $t \in T$ and $v + q\mathcal{L}_t = (\text{EM}(T_\alpha), \text{ME}(T_\beta))$. Also, by assertions $B_r(N)$ and $C_r(N)$, we have

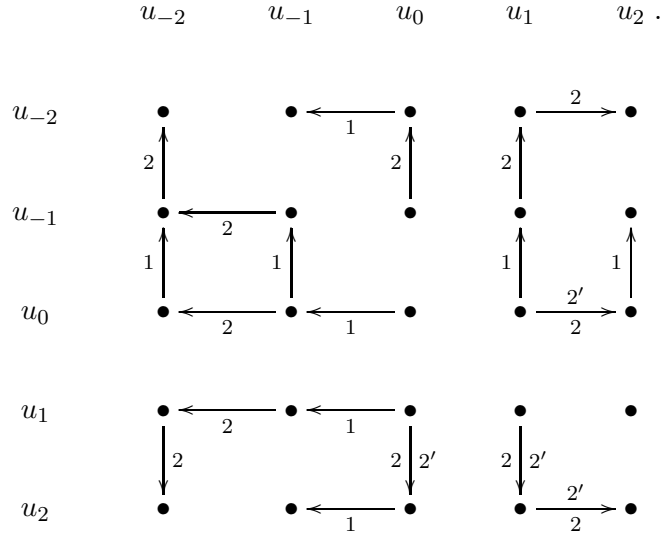
$$\mathcal{B}_t = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(v) + q\mathcal{L}_t \mid l \in \mathbb{Z}_{\geq 0}, i_1, \dots, i_l \in \overline{\mathbb{I}^j}\} \setminus \{0\},$$

and hence,

$$\mathcal{L}_t = \sum_{\substack{l \in \mathbb{Z}_{\geq 0} \\ i_1, \dots, i_l \in \mathbb{I}^J}} \mathbf{A}_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(v).$$

This implies that $(\mathcal{L}(\alpha; \beta), \mathcal{B}(\alpha; \beta))$ equals $(\mathcal{L}_t, \mathcal{B}_t)$, which is a j -crystal basis of $L(\alpha; \beta)$. Moreover, we obtain an isomorphism $\mathcal{B}(\alpha; \beta) \simeq \text{SST}_r(\alpha; \beta)$ of j -crystal graphs. The uniqueness of j -crystal basis of a \mathbf{U}^J -module follows in the same way as that of the ordinary crystal basis of a \mathbf{U} -module. This proves Theorem 7.2.1.

In addition, by assertion $A_r(N)$, we obtain a rule for writing the j -crystal graph of $(\mathcal{L}_r^{\otimes N}, \mathcal{B}_r^{\otimes N})$. For example, the j -crystal graph of $V_2 \otimes V_2$ is as follows:



9.1. The case $r = 2$. Let us prove assertions $A_2(N) - D_2(N)$ by induction on N . The \mathbf{U}_2^J -module structure of V_2 (more generally, the \mathbf{U}_r^J -modules structure of V_r) can be found in [BWW16]. From it, one can easily verify that $V_2 \simeq L(\square; \emptyset) \oplus L(\emptyset; \square)$, and that $(\mathcal{L}_2, \mathcal{B}_2)$ is a j -crystal basis of V_2 whose j -crystal graph is

$$u_{-2} \xleftarrow{2} u_{-1} \xleftarrow{1} u_0 \quad u_1 \xrightarrow{2} u_2.$$

Thus, assertions $A_2(1) - D_2(1)$ are obvious.

Let $N \geq 1$, and assume that assertions $A_2(N) - D_2(N)$ hold. Fix $t \in T$, and write $(\alpha; \beta) = (\alpha_t; \beta_t)$. By $D_2(N)$, we have $|\alpha; \beta| = N$. Let $v \in L(\alpha; \beta)$ be a highest weight vector. In Section 6, we considered $L := L(\alpha; \beta) \otimes V_2 \subset V_2^{\otimes(N+1)}$ and defined five vectors (some of which are equal to 0) $v_0, v_{\pm 1}, v_{\pm 2} \in L$; we see that for $s \in \{0, \pm 1, \pm 2\}$, v_s is nonzero if and only if $(\alpha; \beta)$ is s -addable. Set $S = S(\alpha, \beta) := \{s \in \{0, \pm 1, \pm 2\} \mid (\alpha, \beta) \text{ is } s\text{-addable}\}$. For each $s \in S$, we denote by $(\alpha^s, \beta^s) = (\alpha_t^s, \beta_t^s)$ the double partition obtained from $(\alpha; \beta)$ by adding a box to the $(|s| + 1)$ -st row of α if $s \leq 0$, and to the s -th row of β if $s > 0$.

Proposition 9.1.1. *For each $s \in S$, we have $\mathbf{U}_2^J v_s \simeq L(\alpha^s; \beta^s)$. In particular, assertion $D_2(N + 1)$ holds.*

Proof. By Propositions 6.2.7 and 6.2.11. □

Corollary 9.1.2. *For each double partition (α', β') of $N + 1$, there exists $v_{(\alpha', \beta')} \in \mathcal{L}_2^{\otimes(N+1)}$ such that $\mathbf{U}_2^J v_{(\alpha', \beta')} \simeq L(\alpha'; \beta')$ and $v_{(\alpha', \beta')} + q\mathcal{L}_2^{\otimes(N+1)} = (\text{EM}(T_{\alpha'}), \text{ME}(T_{\beta'}))$.*

Proof. This is easy by the construction of the v_s 's and Proposition 9.1.1. □

Proposition 9.1.3. *Let $s \in S$. Then, $\tilde{f}_{2'}^c(v_s) + q\mathcal{L} \in \mathcal{B} \setminus \{0\}$ if $0 \leq c \leq (\beta^s)_1 - (\beta^s)_2$. Moreover, we have the explicit formula:*

$$\tilde{f}_{2'}^c(v_s) + q\mathcal{L} = \begin{cases} \tilde{f}_{2'}^c(v) \otimes u_0 + q\mathcal{L} & \text{if } s = 0, \\ \tilde{f}_1 \tilde{f}_{2'}^c(v) \otimes u_0 + q\mathcal{L} & \text{if } s = -1, \\ \tilde{f}_{2'}^c(v) \otimes u_1 + q\mathcal{L} & \text{if } s = 1 \text{ and } c \neq (\beta^1)_1 - (\beta^1)_2, \\ \tilde{f}_{2'}^{c-1}(v) \otimes u_2 + q\mathcal{L} & \text{if } s = 1 \text{ and } c = (\beta^1)_1 - (\beta^1)_2, \\ \tilde{f}_1 \tilde{f}_2 \tilde{f}_{2'}^c(v) \otimes u_0 + q\mathcal{L} & \text{if } s = -2, \\ \tilde{f}_{2'}^c(v) \otimes u_2 + q\mathcal{L} & \text{if } s = 2 \text{ and } 0 \leq c \leq (\beta^2)_1 - (\beta^2)_2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The assertions follow from Propositions 6.2.8, 6.2.12, and Corollary 6.2.9. \square

Remark 9.1.4. If we regard $v_s + q\mathcal{L}$, $s \in S$, as an element $\mathbf{s} = (s_1, \dots, s_{N+1}) \in \mathcal{B}_2^{\otimes(N+1)}$, then for $0 \leq s \leq (\beta^s)_1 - (\beta^s)_2$, the $\tilde{f}_{2'}^c(v_s) + q\mathcal{L}$ described in Proposition 9.1.3 is identical to $\tilde{f}_{2'}^c(\mathbf{s})$ defined in Section 8.2.

For $s \in S$, let $L_s = L_{t,s} := \mathbf{U}_2^j v_s$, $\mathcal{L}_s = \mathcal{L}_{t,s} := \mathcal{L} \cap L_s$, and $\mathcal{B}_s = \mathcal{B}_{t,s} := \mathcal{B} \cap (\mathcal{L}_s / q\mathcal{L}_s)$. We will show that $L = \bigoplus_{s \in S} L_s$, $\mathcal{L} = \bigoplus_{s \in S} \mathcal{L}_s$, $\mathcal{B} = \bigsqcup_{s \in S} \mathcal{B}_s$, and that $(\mathcal{L}_s, \mathcal{B}_s)$ is a j -crystal basis of L_s .

Lemma 9.1.5. *We have $\dim L = \sum_{s \in S} \# \text{SST}_2(\alpha^s; \beta^s)$.*

Proof. For a partition λ , we denote by $\text{SST}_l(\lambda)$ the set of semistandard Young tableaux of shape λ in letters $1, \dots, l$. Clearly, we have

$$\dim L = \# \text{SST}_3(\alpha) \cdot \# \text{SST}_2(\beta) \cdot 5.$$

By the Pieri rule for \mathbf{U} , it follows that

$$\begin{aligned} \# \text{SST}_3(\alpha) \cdot 3 &= \sum_{s \in S \cap \mathbb{Z}_{\leq 0}} \# \text{SST}_3(\alpha^s), \\ \# \text{SST}_2(\beta) \cdot 2 &= \sum_{s \in S \cap \mathbb{Z}_{> 0}} \# \text{SST}_2(\beta^s). \end{aligned}$$

Therefore, we see that

$$\begin{aligned} \dim L &= \# \text{SST}_3(\alpha) \cdot 3 \cdot \# \text{SST}_2(\beta) + \# \text{SST}_3(\alpha) \cdot \# \text{SST}_2(\beta) \cdot 2 \\ &= \sum_{s \in S \cap \mathbb{Z}_{\leq 0}} \# \text{SST}_3(\alpha^s) \cdot \# \text{SST}_2(\beta) + \# \text{SST}_3(\alpha) \cdot \sum_{s \in S \cap \mathbb{Z}_{> 0}} \# \text{SST}_2(\beta^s). \end{aligned}$$

Here, notice that $(\alpha^s; \beta^s) = (\alpha^s; \beta)$ if $s \leq 0$, and $(\alpha^s; \beta^s) = (\alpha; \beta^s)$ if $s > 0$. This implies that

$$\begin{aligned} \dim L &= \sum_{s \in S \cap \mathbb{Z}_{\leq 0}} \# \text{SST}_3(\alpha^s) \cdot \# \text{SST}_2(\beta) + \# \text{SST}_3(\alpha) \cdot \sum_{s \in S \cap \mathbb{Z}_{> 0}} \# \text{SST}_2(\beta^s) \\ &= \sum_{s \in S} \# \text{SST}_3(\alpha^s) \cdot \# \text{SST}_2(\beta^s) = \sum_{s \in S} \# \text{SST}_2(\alpha^s; \beta^s), \end{aligned}$$

as desired. \square

Proposition 9.1.6. *The following hold.*

- (1) *For each $s \in S$, the $(\mathcal{L}_s, \mathcal{B}_s)$ is a j -crystal basis of L_s whose j -crystal graph is isomorphic to $\text{SST}_2(\alpha^s; \beta^s)$.*
- (2) *We have $L = \bigoplus_{s \in S} L_s$, $\mathcal{L} = \bigoplus_{s \in S} \mathcal{L}_s$, and $\mathcal{B} = \bigsqcup_{s \in S} \mathcal{B}_s$.*

Proof. (1) Let $s \in S$. By Corollary 9.1.2, we may assume that $v_s + q\mathcal{L} = (\text{EM}(T_{\alpha^s}), \text{ME}(T_{\beta^s}))$. From Proposition 9.1.3, we see that $\tilde{f}_{2'}^c(v_s) \in \mathcal{L}$, $\tilde{f}_{2'}^c(v_s) + q\mathcal{L} \in \mathcal{B} \sqcup \{0\}$, and that $\tilde{f}_{2'}^c(v_s) + q\mathcal{L} \neq 0$ if and only if $0 \leq c \leq (\beta^s)_1 - (\beta^s)_2$. Therefore, we have

$$(14) \quad \begin{aligned} \mathcal{L}_s &\supset \bigoplus_{c=0}^{(\beta^s)_1 - (\beta^s)_2} \sum_{\substack{l \in \mathbb{Z}_{\geq 0} \\ i_1, \dots, i_l \in \{1, 2\}}} \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \tilde{f}_{2'}^c(v_s), \\ \mathcal{B}_s &\supset \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \tilde{f}_{2'}^c(v_s) + q\mathcal{L}_s \mid c, l \in \mathbb{Z}_{\geq 0}, i_1, \dots, i_l \in \{1, 2\}\} \setminus \{0\}. \end{aligned}$$

Also, it is easy to see that $\tilde{f}_{2'}^c(T_{\alpha^s}; T_{\beta^s}) \neq 0$ if and only if $0 \leq c \leq (\beta^s)_1 - (\beta^s)_2$, and that

$$\text{SST}_2(\alpha^s; \beta^s) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \tilde{f}_{2'}^c(T_{\alpha^s}; T_{\beta^s}) \mid c, l \in \mathbb{Z}_{\geq 0}, i_1, \dots, i_l \in \{1, 2\}\} \setminus \{0\}.$$

Therefore, the assignment $0 \neq \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \tilde{f}_{2'}^c(T_{\alpha^s}; T_{\beta^s}) \mapsto \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \tilde{f}_{2'}^c(v_s) + q\mathcal{L}_s$ gives an injection $\text{SST}_2(\alpha^s; \beta^s) \hookrightarrow \mathcal{B}_s$, and hence, $\dim L_s \geq \sharp \text{SST}_2(\alpha^s; \beta^s)$. However, by Lemma 9.1.5, this inequality is indeed an equality, and hence, so are the inclusions \supset in equation (14). In addition, by Proposition 9.1.3, we see that \mathcal{L}_s is closed under $\tilde{f}_{2'}$ and $\tilde{e}_{2'}$. Now, it is easy to check that $(\mathcal{L}_s, \mathcal{B}_s)$ is a \mathcal{J} -crystal basis of L_s such that $\mathcal{B}_s \simeq \text{SST}_2(\alpha^s; \beta^s)$. This proves assertion (1).

(2) Since $L \supset \bigoplus_{s \in S} L_s$ and $\dim L = \sum_{s \in S} \dim L_s$, we have $L = \bigoplus_{s \in S} L_s$. Next, We show that the \mathcal{B}_s 's are pairwise disjoint. Let $b \in \mathcal{B}_s \cap \mathcal{B}_{s'}$ for some $s, s' \in S$. Then, there exist $i_1, \dots, i_k, j_1, \dots, j_l \in \{1, 2\}$ and $c, c' \in \mathbb{Z}_{\geq 0}$ such that

$$\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} \tilde{f}_{2'}^c(v_s) + q\mathcal{L} = b = \tilde{f}_{j_1} \cdots \tilde{f}_{j_l} \tilde{f}_{2'}^{c'}(v_{s'}) + q\mathcal{L}.$$

Equivalently, we have

$$v_s + q\mathcal{L} = \tilde{e}_{2'}^{c'} \tilde{e}_{j_l} \cdots \tilde{e}_{j_1} \tilde{f}_{i_1} \cdots \tilde{f}_{i_k} \tilde{f}_{2'}^c(v_s) + q\mathcal{L}.$$

Since $\mathcal{B}_s \sqcup \{0\}$ is closed under the Kashiwara operators, we have $v_{s'} \in \mathcal{B}_s$, and hence, $s' = s$. Thus, $\mathcal{B} = \bigsqcup_{s \in S} \mathcal{B}_s$. Now, $\mathcal{L} = \bigoplus_{s \in S} \mathcal{L}_s$ is obvious. This completes the proof of the proposition. \square

Recall the irreducible decomposition $V_2^{\otimes N} = \bigoplus_{t \in T} L_t$. Since we took $(\alpha; \beta) = (\alpha_t; \beta_t)$ with $t \in T$ arbitrarily in the second paragraph of this subsection, this proposition implies the equalities

$$\begin{aligned} V_2^{\otimes(N+1)} &= \bigoplus_{t \in T} (L_t \otimes V_2) = \bigoplus_{t \in T} \bigoplus_{s \in S(\alpha_t; \beta_t)} L_{t,s}, \\ \mathcal{L}_2^{\otimes(N+1)} &= \bigoplus_{t \in T} (\mathcal{L}_t \otimes \mathcal{L}_2) = \bigoplus_{t \in T} \bigoplus_{s \in S(\alpha_t; \beta_t)} \mathcal{L}_{t,s}, \\ \mathcal{B}_2^{\otimes(N+1)} &= \bigsqcup_{t \in T} (\mathcal{B}_t \otimes \mathcal{B}_2) = \bigsqcup_{t \in T} \bigsqcup_{s \in S(\alpha_t; \beta_t)} \mathcal{B}_{t,s}. \end{aligned}$$

Moreover, $(\mathcal{L}_{t,s}, \mathcal{B}_{t,s})$ is a \mathcal{J} -crystal basis of $L_{t,s} \simeq L(\alpha_t^s; \beta_t^s)$. This proves assertions $A_2(N+1)$ and $B_2(N+1)$. Now, assertion $C_2(N+1)$ follows from Corollary 9.1.2.

9.2. The case $r \geq 3$. Let $r \geq 3$. We assume that assertions $A_{r'}(N) - D_{r'}(N)$, $N \geq 1$, hold for all $r' < r$, and prove assertions $A_r(N) - D_r(N)$, $N \geq 1$. We proceed by induction on N . The case $N = 1$ is easy (see [BWW16]); indeed, V_r is decomposed as $V_r = \mathbf{U}^J u_0 \oplus \mathbf{U}^J(u_1 - pu_{-1}) \simeq L(\square, \emptyset) \oplus L(\emptyset, \square)$, and the \mathcal{J} -crystal graph is

$$u_{-r} \xleftarrow{r} u_{-(r-1)} \xleftarrow{r-1} \cdots \xleftarrow{1} u_0 \quad u_1 \xrightarrow{2} u_2 \xrightarrow{3} \cdots \xrightarrow{r-1} u_{r-1} \xrightarrow{r} u_r.$$

Let $N \geq 1$, and assume that assertions $A_r(N) - D_r(N)$ hold. Fix $t \in T$, and write $(\alpha; \beta) = (\alpha_t; \beta_t)$. By $D_r(N)$, we have $|\alpha; \beta| = N$. Let $v \in L(\alpha; \beta)$ be a highest weight vector. Set

$S = S(\alpha, \beta) := \{s \in \{-r, \dots, r\} \mid (\alpha, \beta) \text{ is } s\text{-addable}\}$. For each $s \in S$, we denote by $(\alpha^s, \beta^s) = (\alpha_t^s, \beta_t^s)$ the double partition obtained from $(\alpha; \beta)$ by adding a box to the $(|s| + 1)$ -st row of α if $s \leq 0$, and to the s -th row of β if $s > 0$.

Proposition 9.2.1. *Let $r \geq 2$. For each $s \in S$, there exists a highest weight vector $v_s \in \mathcal{L}$ such that $\mathbf{U}^j v_s \simeq L(\alpha^s; \beta^s)$ and*

$$v_s + q\mathcal{L} = \begin{cases} v \otimes u_s & \text{if } s \geq 0, \\ \tilde{f}_1 \tilde{f}_2 \cdots \tilde{f}_{|s|}(v) \otimes u_0 & \text{if } s < 0. \end{cases}$$

In particular, assertion $D_r(N + 1)$ holds.

Proof. See Appendix B.6 □

Corollary 9.2.2. *For each double partition $(\alpha; \beta)$ of N , there exists $v_{(\alpha; \beta)} \in \mathcal{L}_r^{\otimes N}$ such that $\mathbf{U}_2^j v_{(\alpha; \beta)} \simeq L(\alpha; \beta)$, and $v_{(\alpha; \beta)} + q\mathcal{L}_r^{\otimes N} \in \mathcal{B}_r^{\otimes N} = (\text{EM}(T_\alpha), \text{ME}(T_\beta))$.*

Proof. This is easily verified by the construction of v_s and Proposition 9.2.1. □

Lemma 9.2.3. *Let $k \in \{-r, \dots, r\}$ and $i \in \mathbb{I}$. Then, for $b \otimes u_k \in \mathcal{B}$, we have*

$$s_i(b \otimes u_k) = \begin{cases} s_i(b) \otimes u_k & \text{if } k \neq \pm i, \pm(i-1), \\ s_i(\tilde{f}_i(b)) \otimes u_{-i} & \text{if } k = -i \text{ and } \tilde{f}_i(b) \neq 0, \\ s_i(b) \otimes u_{-(i-1)} & \text{if } k = -i \text{ and } \tilde{f}_i(b) = 0, \\ s_i(\tilde{e}_i(b)) \otimes u_{-(i-1)} & \text{if } k = -(i-1) \text{ and } \tilde{e}_i(b) \neq 0, \\ s_i(b) \otimes u_{-i} & \text{if } k = -(i-1) \text{ and } \tilde{e}_i(b) = 0, \\ s_i(\tilde{e}_i(b)) \otimes u_{i-1} & \text{if } k = i-1 \text{ and } \tilde{e}_i(b) \neq 0, \\ s_i(b) \otimes u_i & \text{if } k = i-1 \text{ and } \tilde{e}_i(b) = 0, \\ s_i(\tilde{f}_i(b)) \otimes u_i & \text{if } k = i \text{ and } \tilde{f}_i(b) \neq 0, \\ s_i(b) \otimes u_{i-1} & \text{if } k = i \text{ and } \tilde{f}_i(b) = 0. \end{cases}$$

Proof. This is straightforward by using Proposition 5.2.3. □

Proposition 9.2.4. *Let $b \in \mathcal{B}$ and $c \in \mathbb{Z}_{\geq 0}$. If $\tilde{e}_{r'}(b) = 0$, then we have*

$$\tilde{f}_{r'}^c(b) = \begin{cases} \tilde{f}_{r'}^c(v) \otimes u_s + q\mathcal{L} & \text{if } b = v_s \text{ and } s \in S \setminus \{\pm(r-1), \pm r\}, \\ \tilde{f}_1 \cdots \tilde{f}_{r-1} \tilde{f}_{r'}^c(v) \otimes u_{-(r-1)} + q\mathcal{L} & \text{if } b = v_{-(r-1)}, \\ \tilde{f}_{r'}^c(v) \otimes u_{r-1} + q\mathcal{L} & \text{if } b = v_{r-1} \text{ and } 0 \leq c < (\beta_{r-1})_{r-1} - (\beta_{r-1})_r, \\ \tilde{f}_{r'}^{c-1}(v) \otimes u_r + q\mathcal{L} & \text{if } b = v_{r-1} \text{ and } c = (\beta_{r-1})_{r-1} - (\beta_{r-1})_r, \\ \tilde{f}_1 \cdots \tilde{f}_r \tilde{f}_{r'}^c(v) \otimes u_0 + q\mathcal{L} & \text{if } b = v_{-r}, \\ \tilde{f}_{r'}^c(v) \otimes u_r + q\mathcal{L} & \text{if } b = v_r \text{ and } 0 \leq c \leq (\beta_r)_{r-1} - (\beta_r)_r, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We prove the assertion by induction on r . The case $r = 2$ follows from Proposition 9.1.3. When $r \geq 3$, we use Lemma 7.1.6; we have

$$\tilde{f}_{r'}^c(b) = s_{r-1} s_r \tilde{f}_{(r-1)'}^c s_r s_{r-1}(b).$$

Since $\tilde{f}_{r'}^c(b) = 0$ unless $b = \tilde{f}_{r'}^d(v_s) + q\mathcal{L}$ for some $s \in S$ and $d \in \mathbb{Z}_{\geq 0}$, we may assume that $b = v_s$ for some $s \in S$. In particular, b is identified with $(\text{EM}(T_{\alpha^s}), \text{ME}(T_{\beta^s}))$. Based on this fact, we can compute $s_{r-1} s_r \tilde{f}_{(r-1)'}^c s_r s_{r-1}(b)$ in terms of semistandard double Young tableaux of shape $(\alpha^s; \beta^s)$. Since this calculation is straightforward, we omit the details. □

Lemma 9.2.5. *We have $\dim L = \sum_{s \in S} \# \text{SST}_r(\alpha^s; \beta^s)$.*

Proof. The proof is the same as the proof of Lemma 9.1.5. □

Proposition 9.2.6. *We have $L = \bigoplus_{s \in S} L_s$, $\mathcal{L} = \bigoplus_{s \in S} \mathcal{L}_s$, $\mathcal{B} = \bigsqcup_{s \in S} \mathcal{B}_s$. For each $s \in S$, the $(\mathcal{L}_s, \mathcal{B}_s)$ is a j -crystal basis of L_s whose j -crystal graph is isomorphic to $\text{SST}_r(\alpha^s; \beta^s)$.*

Proof. The proof is the same as the proof of Proposition 9.1.6. \square

Recall the irreducible decomposition $V_r^{\otimes N} = \bigoplus_{t \in T} L_t$. Since we took $(\alpha; \beta) = (\alpha_t; \beta_t)$ with $t \in T$ arbitrarily in the second paragraph of this subsection, this proposition implies the equalities

$$\begin{aligned} V_r^{\otimes(N+1)} &= \bigoplus_{t \in T} (L_t \otimes V_r) = \bigoplus_{t \in T} \bigoplus_{s \in S(\alpha_t; \beta_t)} L_{t,s}, \\ \mathcal{L}_r^{\otimes(N+1)} &= \bigoplus_{t \in T} (\mathcal{L}_t \otimes \mathcal{L}_r) = \bigoplus_{t \in T} \bigoplus_{s \in S(\alpha_t; \beta_t)} \mathcal{L}_{t,s}, \\ \mathcal{B}_r^{\otimes(N+1)} &= \bigsqcup_{t \in T} (\mathcal{B}_t \otimes \mathcal{B}_r) = \bigsqcup_{t \in T} \bigsqcup_{s \in S(\alpha_t; \beta_t)} \mathcal{B}_{t,s}. \end{aligned}$$

Moreover, $(\mathcal{L}_{t,s}, \mathcal{B}_{t,s})$ is a j -crystal basis of $L_{t,s} \simeq L(\alpha_t^s; \beta_t^s)$. This proves assertions $A_r(N+1)$ and $B_r(N+1)$. Now, assertion $C_r(N+1)$ follows from Corollary 9.2.2.

As a byproduct, we obtain the following.

Corollary 9.2.7. *Let $M \in \mathcal{O}_{\text{int}}^j$ be a \mathbf{U}^j -module having a j -crystal basis $(\mathcal{L}, \mathcal{B})$, and let N be a \mathbf{U} -module having a crystal basis $(\mathcal{L}', \mathcal{B}')$. Then, $M \otimes N$ has a j -crystal basis $(\mathcal{L} \otimes \mathcal{L}', \mathcal{B} \otimes \mathcal{B}')$.*

Proof. By Proposition 9.2.6, the assertion holds for $M = L(\alpha; \beta)$ for some double partition $(\alpha; \beta)$, and $N = V_r$. In the general case, M is a direct sum of various $L(\alpha; \beta)$'s, and N is a direct summand of $V_r^{\otimes n}$ for some $n \geq 1$. Therefore, the assertion follows by applying Proposition 9.2.6 repeatedly. \square

10. APPLICATIONS

In this section, we consider how a given \mathbf{U}^j -module decomposes into irreducible modules. By the existence and uniqueness of a j -crystal basis, together with the connectedness (with a single source) of the j -crystal basis of an irreducible \mathbf{U}^j -module, the problem is reduced to determining the highest weight vectors in the j -crystal basis of a given module. We will frequently use results in [Kw09] without mentioning it.

10.1. Irreducible decomposition of $V_r^{\otimes N}$. Let us consider $V_r^{\otimes N}$ and its j -crystal basis $\mathcal{B}_r^{\otimes N}$. The connected components of $\mathcal{B}_r^{\otimes N}$ are in one-to-one correspondence with $\mathbf{s} \in \mathcal{B}_r^{\otimes N}$ such that $\tilde{e}_i(\mathbf{s}) = 0$ for all $i \in \overline{\mathbb{I}}$. Such \mathbf{s} 's are characterized as follows.

Proposition 10.1.1. *Let $\mathbf{s} = (s_1, \dots, s_N) \in \mathcal{B}_r^{\otimes N}$. For each $-r \leq j \leq r$ and $1 \leq n \leq N$, set $c_j^{\leq n}(\mathbf{s}) := \#\{1 \leq m \leq n \mid s_m = j\}$ and $c_j^{\geq n}(\mathbf{s}) := \#\{n \leq m \leq N \mid s_m = j\}$. Then the following are equivalent:*

- (1) $\tilde{e}_i(\mathbf{s}) = 0$ for all $i \in \overline{\mathbb{I}}$.
- (2) $c_0^{\geq n}(\mathbf{s}) \geq c_{-1}^{\geq n}(\mathbf{s})$, $c_{-(j-1)}^{\geq n}(\mathbf{s}) \geq c_{-j}^{\geq n}(\mathbf{s})$, and $c_{j-1}^{\leq n}(\mathbf{s}) \geq c_j^{\leq n}(\mathbf{s})$ for all $1 \leq n \leq N$ and $j \in \mathbb{I} \setminus \{1\}$.

Proof. This follows easily from the j -crystal structure of $\mathcal{B}_r^{\otimes N}$ described at the beginning of Section 8.2. \square

We call an element $\mathbf{s} \in \mathcal{B}_r^{\otimes N}$ satisfying condition (2) of Proposition 10.1.1 a double Yamanouchi word, since \mathbf{s} is a Yamanouchi word when we read only letters $1, 2, \dots, r$ and so is \mathbf{s}^{rev} when we read only letters $0, -1, \dots, -r$ and then ignore negative signs.

Remark 10.1.2. What we call a Yamanouchi word is called a lattice permutation in [Kw09]. For a partition λ of length r , we denote by $\text{Yam}(\lambda)$ the set of Yamanouchi words in letters $1, \dots, r$ of shape λ , that is, the number of appearances of i in the word equals λ_i for all $i \in \{1, \dots, r\}$. By the Robinson-Schensted correspondence, one has $\#\text{Yam}(\lambda) = \#\text{ST}(\lambda)$.

Proposition 10.1.3. *Let $\mathbf{s} \in \mathcal{B}_r^{\otimes N}$ be a double Yamanouchi word. Then, the connected component of $\mathcal{B}_r^{\otimes N}$ containing \mathbf{s} is isomorphic to $\text{SST}_r(\alpha; \beta)$, where $\alpha = (\alpha_1, \dots, \alpha_{r+1})$ and $\beta = (\beta_1, \dots, \beta_r)$ are partitions given by*

$$(15) \quad \alpha_i = \#\{m \mid s_m = -(i-1)\}, \quad \beta_i = \#\{m \mid s_m = i\}.$$

Proof. By the complete reducibility of $V_r^{\otimes N}$, the connected component of $\mathcal{B}_r^{\otimes N}$ is isomorphic to $\text{SST}_r(\alpha; \beta)$ for some double partition $(\alpha; \beta)$ of size N . Since $\tilde{e}_i(\mathbf{s}) = 0$ for all $i \in \overline{\mathbb{I}}$, we may identify \mathbf{s} with $(\text{EM}(T_\alpha), \text{ME}(T_\beta))$, which satisfies condition (15). This proves the proposition. \square

We denote by $\text{Yam}(\alpha; \beta)$ the set of double Yamanouchi words in $\mathcal{B}_r^{\otimes |\alpha; \beta|}$ satisfying condition (15), and call each element in $\text{Yam}(\alpha; \beta)$ a Yamanouchi word of shape $(\alpha; \beta)$.

Definition 10.1.4. (1) A semistandard Young tableau T of shape α is said to be standard if $\text{Im } T = \{1, \dots, |\alpha|\}$. We denote by $\text{ST}(\alpha)$ the set of standard Young tableaux of shape α .

(2) A semistandard double Young tableau (T_1, T_2) of shape $(\alpha; \beta)$ is said to be standard if $\text{Im } T_1 \sqcup \text{Im } T_2 = \{1, \dots, |\alpha; \beta|\}$. We denote by $\text{ST}(\alpha; \beta)$ the set of standard double Young tableaux of shape $(\alpha; \beta)$.

Let $(\alpha; \beta)$ be a double partition and $(T_1; T_2) \in \text{ST}(\alpha; \beta)$. We write $\text{Im } T_1 = \{p_1, \dots, p_{|\alpha|}\}$ and $\text{Im } T_2 = \{q_1, \dots, q_{|\beta|}\}$, with $p_1 < \dots < p_{|\alpha|}$, $q_1 < \dots < q_{|\beta|}$. Let T'_1 denote the standard Young tableau of shape α obtained from T_1 by replacing each p_i with i . Define T'_2 similarly. Then, the map $\text{ST}(\alpha; \beta) \rightarrow \text{ST}(\alpha) \times \text{ST}(\beta) \times \{(q_1, \dots, q_{|\beta|}) \mid 1 \leq q_1 < \dots < q_{|\beta|} \leq |\alpha; \beta|\}$ defined by $(T_1, T_2) \mapsto (T'_1, T'_2, (q_1, \dots, q_{|\beta|}))$ is a bijection.

Theorem 10.1.5. *Let $(\alpha; \beta)$ be a double partition of N of length $(r+1; r)$. Then, the multiplicity of the irreducible component of $V_r^{\otimes N}$ isomorphic to $L(\alpha; \beta)$ is equal to $\#\text{ST}(\alpha; \beta)$. Namely, we have an isomorphism*

$$V_r^{\otimes N} \simeq \bigoplus_{(\alpha; \beta)} L(\alpha; \beta)^{\oplus (\#\text{ST}(\alpha; \beta))}$$

of \mathbf{U}^j -modules, where $(\alpha; \beta)$ runs over all double partitions of N of length $(r+1; r)$.

Proof. By Proposition 10.1.3, the multiplicity of the irreducible component of $V_r^{\otimes N}$ isomorphic to $L(\alpha; \beta)$ is equal to $\#\text{Yam}(\alpha; \beta)$. Here, the set $\text{Yam}(\alpha; \beta)$ is in one-to-one correspondence with $\text{Yam}(\alpha) \times \text{Yam}(\beta) \times \{(q_1, \dots, q_{|\beta|}) \mid 1 \leq q_1 \leq \dots \leq q_{|\beta|} \leq N\}$ under the assignment

$$\mathbf{s} = (s_1, \dots, s_N) \mapsto \left((|s_{p_{|\alpha|}}|, \dots, |s_{p_1}|), (s_{q_1}, \dots, s_{q_{|\beta|}}), (q_1, \dots, q_{|\beta|}) \right),$$

where $s_{p_1}, \dots, s_{p_{|\alpha|}} \leq 0$ with $p_1 < \dots < p_{|\alpha|}$, and $s_{q_1}, \dots, s_{q_{|\beta|}} \geq 1$ with $q_1 < \dots < q_{|\beta|}$. From this and the bijection $\text{ST}(\alpha; \beta) \rightarrow \text{ST}(\alpha) \times \text{ST}(\beta) \times \{(q_1, \dots, q_{|\beta|}) \mid 1 \leq q_1 \leq \dots \leq q_{|\beta|} \leq |\alpha; \beta|\}$, we obtain

$$\begin{aligned} \#\text{Yam}(\alpha; \beta) &= \#\text{Yam}(\alpha) \cdot \#\text{Yam}(\beta) \cdot \binom{N}{|\beta|} \\ &= \#\text{ST}(\alpha) \cdot \#\text{ST}(\beta) \cdot \binom{N}{|\beta|} \quad (\text{By Remark 10.1.2}) \\ &= \#\text{ST}(\alpha; \beta), \end{aligned}$$

as desired. This proves the theorem. \square

By the double centralizer property for \mathbf{U}^j and the Hecke algebra $\mathcal{H}(W_d)$ of type B_d with unequal parameters (p, q) on $V_r^{\otimes d}$ for $r \geq d$ ([BWW16]), we have an irreducible decomposition

$$(16) \quad V_r^{\otimes d} \simeq \bigoplus_{(\alpha; \beta)} L(\alpha; \beta) \boxtimes V(\alpha; \beta)$$

as a $\mathbf{U}^J\text{-}\mathcal{H}(B_d)$ -bimodule, where $(\alpha; \beta)$ runs over all double partitions of N of length $(r+1; r)$, and $V(\alpha; \beta)$ is an irreducible $\mathcal{H}(W_d)$ -module. According to [H74], the irreducible $\mathcal{H}(W_d)$ -modules are classified by the double partitions of size d . Hoefsmit constructed the irreducible modules by giving the representation matrices for the generators of $\mathcal{H}(W_d)$ explicitly. Later, Dipper and James [DJ92] realized the irreducible $\mathcal{H}(W_d)$ -modules $S^{\alpha, \beta}$ as ideals of $\mathcal{H}(W_d)$. In Appendix A, we prove that $V(\alpha; \beta) \simeq S^{\alpha, \beta}$.

10.2. Littlewood-Richardson rule for \mathbf{U}^J . In this section, we consider the irreducible decomposition of $L(\alpha; \beta) \otimes L(\lambda)$, where $L(\lambda)$ denotes the irreducible highest weight \mathbf{U} -module with highest weight λ . In terms of J -crystal bases, we will determine the double Yamanouchi words in $\mathcal{B}(\alpha; \beta) \otimes \mathcal{B}(\lambda) \in \mathcal{B}_r^{\otimes |\alpha; \beta| + |\lambda|}$; here $\mathcal{B}(\lambda) \simeq \text{SST}(\lambda)$ denotes the crystal basis of $L(\lambda)$ embedded in $\mathcal{B}_r^{\otimes |\lambda|}$ by the Middle-Eastern reading. Let $\text{LR}_{(\alpha; \beta), \lambda}^{(\alpha'; \beta')}(r)$ denote the multiplicity of $L(\alpha'; \beta')$ in $L(\alpha; \beta) \otimes L(\lambda)$; clearly, it is equal to the number of double Yamanouchi words in $\mathcal{B}(\alpha; \beta) \otimes \mathcal{B}(\lambda)$ of shape $(\alpha'; \beta')$.

Let us briefly recall the Littlewood-Richardson rule for ordinary crystal bases in type A . Let $\text{LR}_{\mu, \nu}^{\lambda}(2r+1)$ denote the multiplicity of $L(\lambda)$ in $L(\mu) \otimes L(\nu)$. A semistandard tableau T of shape λ/μ is called a Littlewood-Richardson tableau of shape λ/μ with content ν if T contains ν_i i 's, and if $\text{ME}(T)$ is a Yamanouchi word ([Kw09]). Hence $\text{LR}_{\mu, \nu}^{\lambda}(2r+1)$ equals the number of Littlewood-Richardson tableaux of shape λ/μ with content ν in $2r+1$ letters. Also, it is known that the multiplicity of $\mathcal{B}(\nu)$ in $\mathcal{B}(\lambda/\mu)$ is equal to $\text{LR}_{\mu, \nu}^{\lambda}(2r+1)$.

Theorem 10.2.1. *Let $(\alpha; \beta)$, $(\alpha'; \beta')$ be double partitions of length $(r+1; r)$, and λ a partition of length $2r+1$. Then, we have*

$$\text{LR}_{(\alpha; \beta), \lambda}^{(\alpha'; \beta')}(r) = \sum_{\substack{\mu \subset \lambda \\ \ell(\mu) \leq r+1}} \sum_{\nu} \text{LR}_{\mu, \alpha}^{\alpha'}(r+1) \text{LR}_{\mu, \nu}^{\lambda}(r) \text{LR}_{\beta, \nu}^{\beta'}(r),$$

where ν runs over all partitions of size $|\lambda/\mu|$.

Proof. Let $(T_1; T_2) \in \mathcal{B}(\alpha; \beta)$ and $T \in \mathcal{B}(\lambda)$. If we read only letters ≤ 0 in T , then it is also a semistandard tableau T' of shape, say $\mu \subset \lambda$. Since there are $r+1$ kinds of letters ≤ 0 , we have $\ell(\mu) \leq r+1$. Suppose that $(T_1; T_2) \otimes T$ is a double Yamanouchi word of shape $(\alpha'; \beta')$. By the definition of double Yamanouchi words, $(\text{ME}(T_2), \text{ME}(T/T'))$ is a Yamanouchi word of shape β' in letters $1, \dots, r$, and $(\text{EM}(T_1), \text{ME}(T'))^{\text{rev}} = (\text{EM}(T'), \text{ME}(T_1))$ is a Yamanouchi word of shape α' in letters $0, 1, \dots, r$ if we ignore negative signs. In addition, by Proposition 5.2.5, we have $\tilde{F}_{-(i-\frac{1}{2})}(T') = 0$ for all $i \in \mathbb{P}$. This implies that $\text{EM}(T')$ is a Yamanouchi word of shape μ if we ignore negative signs, and that T' is determined uniquely by μ and this condition; hence, we write $T'(\mu) = T'$. With this notation, for an arbitrary partition $\mu \subset \lambda$ of length $\leq r+1$, let $Y(\mu)$ be the number of (T_2, T) such that $(\text{ME}(T_2), \text{ME}(T/T'(\mu)))$ is a Yamanouchi word of shape β' in letters $1, \dots, r$, and $Z(\mu)$ the number of T_1 such that $(\text{EM}(T'(\mu)), \text{ME}(T_1))$ is a Yamanouchi word of shape α' in letters $0, 1, \dots, r$ if we ignore negative signs. Then, by the above, we obtain

$$\text{LR}_{(\alpha; \beta), \lambda}^{(\alpha'; \beta')}(r) = \sum_{\substack{\mu \subset \lambda \\ \ell(\mu) \leq r+1}} Y(\mu) \cdot Z(\mu);$$

here, $Y(\mu)$ is equal to the cardinality of $\text{Yam}(\beta') \cap (\mathcal{B}(\beta) \otimes \mathcal{B}_r(\lambda/\mu))$, where $\mathcal{B}_r(\lambda/\mu)$ denotes the set of semistandard tableaux of shape λ/μ in letters $1, \dots, r$. Therefore, we see that $Y(\mu) = \sum_{\nu \vdash |\lambda/\mu|} \text{LR}_{\beta, \nu}^{\beta'}(r) \cdot \text{LR}_{\mu, \nu}^{\lambda}(r)$ by the Littlewood-Richardson rule for ordinary crystal bases in type A .

In order to compute $Z(\mu)$, let us count the number $Z'(\mu)$ of Yamanouchi words in $\mathcal{B}(\mu) \otimes \mathcal{B}(\alpha)$ of shape α' in letters $0, 1, \dots, r$. By the tensor product rule for ordinary crystal bases, if $T_3 \otimes T_4 \in \mathcal{B}(\mu) \otimes \mathcal{B}(\alpha)$ is a Yamanouchi word, then so is T_3 . Since $\text{EM}(T'(\mu))$ is a Yamanouchi word of

shape μ in letters $0, 1, \dots, r$ if we ignore negative signs, $Z(\mu)$ is equal to $Z'(\mu)$, which, in turn, equals $\text{LR}_{\mu, \alpha}^{\alpha'}(r+1)$ by the Littlewood-Richardson rule for ordinary crystal basis in type A .

Summarizing, we conclude that

$$\text{LR}_{(\alpha; \beta), \lambda}^{(\alpha'; \beta')} = \sum_{\substack{\mu \subset \lambda \\ \ell(\mu) \leq r+1}} \sum_{\nu \vdash |\lambda/\mu|} \text{LR}_{\beta, \nu}^{\beta'}(r) \cdot \text{LR}_{\mu, \nu}^{\lambda}(r) \cdot \text{LR}_{\mu, \alpha}^{\alpha'}(r+1),$$

as desired. This proves the theorem. \square

In particular, if we take $(\alpha; \beta)$ to be $(\emptyset; \emptyset)$, then the tensor product module $L(\emptyset; \emptyset) \otimes L(\lambda)$ is just $L(\lambda)$ regarded as a \mathbf{U}^j -module. Hence, Theorem 10.2.1 gives the branching rule for \mathbf{U} -modules restricted to \mathbf{U}^j :

Corollary 10.2.2. *The multiplicity of $L(\alpha'; \beta')$ in $L(\lambda)$ is equal to*

$$\text{LR}_{(\emptyset; \emptyset), \lambda}^{(\alpha'; \beta')} = \text{LR}_{\alpha', \beta'}^{\lambda}.$$

Proof. By Theorem 10.2.1, we have

$$\text{LR}_{(\emptyset; \emptyset), \lambda}^{(\alpha'; \beta')} = \sum_{\mu, \nu} \text{LR}_{\mu, \emptyset}^{\alpha'}(r+1) \text{LR}_{\mu, \nu}^{\lambda}(r) \text{LR}_{\emptyset, \nu}^{\beta'}(r).$$

However, $\text{LR}_{\mu, \emptyset}^{\alpha'}(r+1) = \delta_{\mu, \alpha'}$ and $\text{LR}_{\emptyset, \nu}^{\beta'}(r) = \delta_{\nu, \beta'}$. This proves the corollary. \square

APPENDIX A. IRREDUCIBLE DECOMPOSITION OF $V_r^{\otimes d}$ AS A $\mathbf{U}^j\text{-}\mathcal{H}(W_d)$ -BIMODULE

A.1. The action of $\mathcal{H}(W_d)$ on $V_r^{\otimes d}$. Let $\mathcal{H}(W_d)$ be the $\mathbb{Q}(p, q)$ -algebra generated by H_i , $0 \leq i \leq d-1$, subject to the relations:

$$\begin{aligned} H_0^2 &= (p^{-1} - p)H_0 + 1, \\ H_i^2 &= (q^{-1} - q)H_i + 1 \quad \text{if } i \neq 0, \\ H_0H_1H_0H_1 &= H_1H_0H_1H_0, \\ H_iH_{i+1}H_i &= H_{i+1}H_iH_{i+1} \quad \text{if } 1 \leq i \leq n-2, \\ H_iH_j &= H_jH_i \quad \text{if } |i-j| > 1. \end{aligned}$$

Let W_d denote the Weyl group of type B_d with simple reflections s_0, s_1, \dots, s_{d-1} . We denote by \mathfrak{S}_d and $\mathfrak{S}_{a, d-a}$ the subgroup of W_d generated by s_i , $i \neq 0$, and s_i , $i \neq 0, a$, respectively. Let $\mathcal{H}(\mathfrak{S}_d)$ and $\mathcal{H}(\mathfrak{S}_{a, d-a})$ be the subalgebra of $\mathcal{H}(W_d)$ generated by H_i , $i \neq 0$, and H_i , $i \neq 0, a$, respectively.

Following [BWW16], let us recall the action of $\mathcal{H}(W_d)$ on $V_r^{\otimes d}$. For a map $f : \{1, \dots, d\} \rightarrow \{-r, -r+1, \dots, r\}$, we set $M_f := u_{f(1)} \otimes \dots \otimes u_{f(d)} \in V_r^{\otimes d}$. The Weyl group W_d acts on the set of maps from $\{1, \dots, d\}$ to $\{-r, -r+1, \dots, r\}$ by:

$$\begin{aligned} (f \cdot s_j)(i) &= \begin{cases} f(j+1) & \text{if } i = j, \\ f(j) & \text{if } i = j+1, \\ f(i) & \text{otherwise,} \end{cases} \\ (f \cdot s_0)(i) &= \begin{cases} -f(1) & \text{if } i = 1, \\ f(i) & \text{otherwise.} \end{cases} \end{aligned}$$

Then the Hecke algebra $\mathcal{H}(W_d)$ acts on $V_r^{\otimes d}$ by:

$$\begin{aligned} M_f \cdot H_j &= \begin{cases} q^{-1}M_f & \text{if } f(i) = f(i+1), \\ M_{f \cdot s_j} & \text{if } f(i) < f(i+1), \\ M_{f \cdot s_j} + (q^{-1} - q)M_f & \text{if } f(i) > f(i+1), \end{cases} \\ M_f \cdot H_0 &= \begin{cases} p^{-1}M_f & \text{if } f(1) = 0, \\ M_{f \cdot s_0} & \text{if } f(1) > 0, \\ M_{f \cdot s_0} + (p^{-1} - p)M_f & \text{if } f(1) < 0. \end{cases} \end{aligned}$$

A.2. Irreducible $\mathcal{H}(\mathfrak{S}_d)$ -modules. Let us recall from [G86] how to construct the irreducible $\mathcal{H}(\mathfrak{S}_d)$ -modules. Note that our normalization of the generators of the Hecke algebra differs from that in [G86]; because of this, we construct right $\mathcal{H}(\mathfrak{S}_d)$ -modules, while Gyoja treated left $\mathcal{H}(\mathfrak{S}_d)$ -modules. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0)$ be a partition of d and λ' its transposed partition. Let $T_+(\lambda)$ be the standard tableau of shape λ defined by $T_+(i, j) = \lambda_1 + \dots + \lambda_{i-1} + j$, and $T_-(\lambda)$ the standard tableau of shape λ defined by $T_-(i, j) = \lambda'_1 + \dots + \lambda'_{j-1} + i$. Also, let I_+ be the set of those s_i which preserves each row of T_+ , I_- the set of those s_i which preserves each column of T_- , \mathfrak{S}_\pm the subgroup of \mathfrak{S}_d generated by I_\pm , and $\mathcal{H}(\mathfrak{S}_\pm)$ the subalgebra of $\mathcal{H}(\mathfrak{S}_d)$ corresponding to \mathfrak{S}_\pm . Set

$$e_+ := \sum_{x \in \mathfrak{S}_+} q^{-\ell(x)} H_x, \quad e_- := \sum_{y \in \mathfrak{S}_-} (-q)^{\ell(y)} H_y.$$

Let $[+, -] \in \mathfrak{S}_d$ be the unique element such that $T_+ \cdot [+, -] = T_-$. Then, for each $x \in \mathfrak{S}_+$ and $y \in \mathfrak{S}_-$, one has $\ell(x[+, -]y) = \ell(x) + \ell([+, -]) + \ell(y)$. By [G86, Section 2], the following holds.

Theorem A.2.1 ([G86]). *The right ideal S^λ of $\mathcal{H}(\mathfrak{S}_d)$ generated by $e_+ H_{[+, -]} e_-$ is an irreducible $\mathcal{H}(\mathfrak{S}_d)$ -module. Moreover, the set $\{S^\lambda \mid \lambda \vdash d\}$ provides a complete list of nonisomorphic irreducible $\mathcal{H}(\mathfrak{S}_d)$ -modules.*

By this theorem, we can realize each S^λ , $\lambda \vdash d$, as a submodule of $V_r^{\otimes d}$. We define a map f_λ by: $f_\lambda(i) = j$ if $\lambda_{j-1} < i \leq \lambda_j$. It is easy to verify that the $\mathcal{H}(\mathfrak{S}_d)$ -submodule generated by M_{f_λ} is isomorphic to $e_+ \mathcal{H}(\mathfrak{S}_d)$. Therefore, the $\mathcal{H}(\mathfrak{S}_d)$ -submodule generated by $M_{\lambda,+} := M_{f_\lambda} \cdot (H_{[+, -]} e_-)$ is isomorphic to S^λ . Since $\ell(x[+, -]y) = \ell(x) + \ell([+, -]) + \ell(y)$ for all $x \in \mathfrak{S}_+$ and $y \in \mathfrak{S}_-$, we see that

$$M_{\lambda,+} = \sum_{y \in \mathfrak{S}_-} (-q)^{\ell(y)} M_{f_\lambda \cdot [+, -]y}.$$

Also, by the definitions of f_λ and $[+, -]$, it follows that

$$M_{f_\lambda \cdot [+, -]} = (u_1 \otimes u_2 \otimes \dots \otimes u_{\lambda'_1}) \otimes (u_1 \otimes u_2 \otimes \dots \otimes u_{\lambda'_2}) \otimes \dots \otimes (u_1 \otimes u_2 \otimes \dots \otimes u_{\lambda'_k}).$$

These imply that $M_{\lambda,+} \in M_{f_\lambda \cdot [+, -]} + q\mathcal{L}_r^{\otimes d}$.

By the quantum Schur-Weyl duality of type A , the irreducible $\mathcal{H}(\mathfrak{S}_d)$ -module $M_{\lambda,+} \mathcal{H}(\mathfrak{S}_d) \simeq S^\lambda$ is contained in the direct sum of some copies of the irreducible highest weight \mathbf{U} -module with highest weight corresponding to a partition, say μ . Applying Kashiwara operators \tilde{E}_i 's on $M_{\lambda,+}$ repeatedly, one can easily verify that $\mu = \lambda$.

Exchanging the roles of H_i and H_i^{-1} , we obtain $M_{\lambda,-} \in V_r^{\otimes d}$ such that

$$\begin{aligned} M_{\lambda,-} &\in (u_{-\lambda'_1} \otimes \dots \otimes u_{-2} \otimes u_{-1}) \otimes (u_{-\lambda'_2} \otimes \dots \otimes u_{-2} \otimes u_{-1}) \otimes \dots \\ &\quad \otimes (u_{-\lambda'_k} \otimes \dots \otimes u_{-2} \otimes u_{-1}) + q\mathcal{L}_r^{\otimes d} \end{aligned}$$

and $M_{\lambda,-} \mathcal{H}(\mathfrak{S}_d) \simeq S^\lambda$.

A.3. Irreducible $\mathcal{H}(W_d)$ -modules. In this subsection, we construct the irreducible $\mathcal{H}(W_d)$ -modules following [DJ92]. For $1 \leq i < j \leq d-1$, we set

$$s_{i,j} := s_i s_{i+1} \cdots s_{j-1}, \quad s_{j,i} := s_{i,j}^{-1}.$$

Fix two nonnegative integers a, b such that $a + b = d$, and set $w_{a,b} := (s_{d,1})^b \in \mathfrak{S}_d$. Also, we define $v_{a,b} \in \mathcal{H}(W_d)$ by

$$v_{a,b} := \prod_{i=1}^a (p + H_{s_{i,1}} H_0 H_{s_{1,i}}) H_{w_{a,b}} \prod_{j=1}^b (1 - p H_{s_{j,1}} H_0 H_{s_{1,j}}).$$

Let $\lambda \vdash a$ and $\mu \vdash b$. By Appendix A.2, one can construct the irreducible $\mathcal{H}_{\mathfrak{S}_a}$ -module S^λ in the subalgebra of $\mathcal{H}(W_d)$ generated by H_1, \dots, H_{a-1} , and the irreducible $\mathcal{H}_{\mathfrak{S}_b}$ -module S^μ in the subalgebra generated by H_{a+1}, \dots, H_{n-1} . It follows that $S^\lambda \cdot S^\mu \subset \mathcal{H}(\mathfrak{S}_{a,b})$. Set

$$S^{\lambda,\mu} := S^\lambda \cdot S^\mu \cdot v_{a,b} \mathcal{H}(W_d) = v_{a,b} S^\mu \cdot S^\lambda \mathcal{H}(W_d).$$

Theorem A.3.1 ([DJ92]). *The set $\{S^{\lambda,\mu} \mid 0 \leq a \leq d, \lambda \vdash a, \mu \vdash d-a\}$ provides a complete list of nonisomorphic irreducible $\mathcal{H}(W_d)$ -modules.*

Let us find a good generator of $S^{\lambda,\mu}$ in $V_r^{\otimes d}$. Define a map $f_{\lambda,\mu}$ by:

$$f_{\lambda,\mu}(i) = \begin{cases} f_\lambda(i) & \text{if } 1 \leq i \leq a, \\ f_\mu(i-a) & \text{if } a+1 \leq i \leq d. \end{cases}$$

By Appendix A.2, we have

$$S^\lambda \cdot S^\mu \simeq M_{f_{\lambda,\mu}} S^\lambda \cdot S^\mu \subset M_{f_{\lambda,\mu}} \mathcal{H}(\mathfrak{S}_d),$$

and hence,

$$S^{\lambda,\mu} \simeq M_{f_{\lambda,\mu}} S^\lambda \cdot S^\mu v_{a,b} \mathcal{H}(W_d) = M_{f_{\lambda,\mu}} v_{a,b} S^\mu \cdot S^\lambda \mathcal{H}(W_d).$$

Also, we see that

$$M_{f_{\lambda,\mu}} v_{a,b} \in u_{f_\mu(1)} \otimes \cdots \otimes u_{f_\mu(b)} \otimes u_{-f_\lambda(1)} \otimes \cdots \otimes u_{-f_\lambda(a)} + p \mathcal{L}_r^{\otimes d}.$$

Therefore, $M_{f_{\lambda,\mu}} v_{a,b} S^\mu \cdot S^\lambda \mathcal{H}(W_d)$ is generated by $M_{\mu,+} \otimes M_{\lambda,-}$, which is of the form

$$\begin{aligned} M_{\mu,+} \otimes M_{\lambda,-} &\in (u_1 \otimes \cdots \otimes u_{\mu'_1}) \otimes (u_1 \otimes \cdots \otimes u_{\mu'_2}) \otimes \cdots \otimes (u_1 \otimes \cdots \otimes u_{\mu'_l}) \\ &\otimes (u_{-\lambda'_1} \otimes \cdots \otimes u_{-1}) \otimes (u_{-\lambda'_2} \otimes \cdots \otimes u_{-1}) \otimes \cdots \otimes (u_{-\lambda'_k} \otimes \cdots \otimes u_{-1}) + q \mathcal{L}_r^{\otimes d}. \end{aligned}$$

By the quantum Schur-Weyl duality of type B , the irreducible $\mathcal{H}(W_d)$ -modules $M_{\mu,+} \otimes M_{\lambda,-} \mathcal{H}(W_d) \simeq S^{\lambda,\mu}$ is contained in the direct sum of some copies of the irreducible highest weight \mathbf{U}^j -module $L(\alpha; \beta)$ for some double partition $(\alpha; \beta)$. By the descriptions of $M_{\mu,+} + q \mathcal{L}_r^{\otimes a}$ and $M_{\lambda,-} + q \mathcal{L}_r^{\otimes b}$, it is clear that

$$\begin{aligned} \tilde{e}_1^{\lambda_1} \cdots \tilde{e}_r^{\lambda_r} (M_{\mu,+} \otimes M_{\lambda,-} + q \mathcal{L}_r^{\otimes d}) &\in \mathcal{B}_r^{\otimes d} \\ &= (u_1 \otimes \cdots \otimes u_{\mu'_1}) \otimes (u_1 \otimes \cdots \otimes u_{\mu'_2}) \otimes \cdots \otimes (u_1 \otimes \cdots \otimes u_{\mu'_l}) \\ &\otimes (u_{-(\lambda'_1-1)} \otimes \cdots \otimes u_0) \otimes (u_{-(\lambda'_2-1)} \otimes \cdots \otimes u_0) \otimes \cdots \otimes (u_{-(\lambda'_k-1)} \otimes \cdots \otimes u_0) + q \mathcal{L}_r^{\otimes d} \end{aligned}$$

is a double Yamanouchi word of shape $(\lambda; \mu)$, and hence, we conclude that $L(\alpha; \beta) \simeq L(\lambda; \mu)$.

Theorem A.3.2. *Let $r \geq d$. As a \mathbf{U}^j - $\mathcal{H}(W_d)$ -bimodule, $V_r^{\otimes d}$ is decomposed as follows:*

$$V_r^{\otimes d} \simeq \bigoplus_{(\alpha; \beta)} L(\alpha; \beta) \otimes S^{\alpha, \beta},$$

where $(\alpha; \beta)$ runs over all the double partitions of d .

This theorem implies that the functor F , from the category of $\mathcal{H}(W_d)$ -modules to $\mathcal{O}_{\text{int}}^j$, defined by $F(M) := V_r^{\otimes d} \otimes_{\mathcal{H}(W_d)} M$ gives a category equivalence which maps $(S^{\alpha, \beta})^{\text{op}}$ to $L(\alpha; \beta)$ for all double partitions $(\alpha; \beta)$ of d .

APPENDIX B.

B.1. Proof of Proposition 6.1.1. Let M be a \mathbf{U}^j -module and $m \in M_{a,b,n} \setminus \{0\}$. Set $h'_1 := h_1 + \frac{p^{-1}qk_1^{-1}}{q-q^{-1}}$.

Lemma B.1.1. *We have the following:*

- (1) $[h'_1, f_2]_1 = q^2 f'_2$.
- (2) $[h'_1, f'_2]_1 = q^2 f_2$.
- (3) $[h'_1, f'_2]_{-1} = -p \left(q^{-3} \overline{f'_2} - [2] f'_2 - f_2 \left(q^{-1}(q - q^{-1}) \overline{h'_1} + [2] p^{-1} q^{-1} k_1^{-1} \right) \right) k_1$.

Proof. This is easy and straightforward. \square

Since h_1 and k_1 act on m as scalar multiplication, so does h'_1 ; explicitly, we have $h'_1 m = h'_1(a, b, n)m$, where

$$h'_1(a, b, n) := [n+1][b-n]\{a-b-n-1\} - q[n][b-n+1]\{a-b-n\} + \frac{p^{-1}q^{-a+3n+1}}{q-q^{-1}}.$$

By Lemma B.1.1, we have

$$\begin{aligned} h'_1 \overline{f'_2} m &= q h'_1(a, b, n) \overline{f'_2} m + q^2 f_2 m, \\ h'_1 f_2 m &= q h'_1(a, b, n) f_2 m + q^2 f'_2 m, \\ h'_1 f'_2 m &= q^{-1} h'_1(a, b, n) f'_2 m - p \left(q^{-3} \overline{f'_2} - [2] f'_2 - f_2 \left(q^{-1}(q - q^{-1}) \overline{h'_1(a, b, n)} + [2] p^{-1} q^{-1} q^{-a+3n} \right) \right) q^{a-3n} \\ &= -p q^{a-3n-3} \overline{f'_2} m + p q^{a-3n} \left(q^{-1}(q - q^{-1}) \overline{h'_1(a, b, n)} + [2] p^{-1} q^{-a+3n-1} \right) f_2 m \\ &\quad + (q^{-1} h'_1(a, b, n) + p q^{a-3n} [2]) f'_2 m. \end{aligned}$$

Therefore, h'_1 defines a linear endomorphism on the vector space spanned by $\{\overline{f'_2} m, f_2 m, f'_2 m\}$ whose representation matrix is

$$(17) \quad \begin{pmatrix} q h'_1(a, b, n) & 0 & -p q^{a-3n-3} \\ q^2 & q h'_1(a, b, n) & p q^{a-3n-1} (q - q^{-1}) \overline{h'_1(a, b, n)} + q^{-1} [2] \\ 0 & q^2 & q^{-1} h'_1(a, b, n) + p q^{a-3n} [2] \end{pmatrix}.$$

Hence, in order to prove Proposition 6.1.1, it suffices to show that the following three vectors

$$\begin{pmatrix} q^{b-n-1} \\ p q^{a-b} - p^{-1} q^{-a+b} \\ -q^{-b+n+1} \end{pmatrix}, \quad \begin{pmatrix} p q^{a-b-n-2} \\ -(q^{b+1} + q^{-b-1}) \\ p^{-1} q^{-a+b+n+2} \end{pmatrix}, \quad \begin{pmatrix} q^{-n-2} \\ p q^{a-2b-1} - p^{-1} q^{-a+2b+1} \\ -q^{n+2} \end{pmatrix}$$

are eigenvectors of the matrix (17) with eigenvalues $h'_1(a+1, b+1, n)$, $h'_1(a+1, b, n)$, and $h'_1(a-2, b-1, n-1)$, respectively. This can be checked by using a computer, or possibly by direct calculation.

B.2. Proof of Proposition 6.1.2.

Lemma B.2.1. *It holds that*

$$f'_2 f_2 = q^{-1} f_2 f'_2, \quad f_2 \overline{f'_2} = q^{-1} \overline{f'_2} f_2, \quad f'_2 \overline{f'_2} = \overline{f'_2} f_2 - (q - q^{-1}) f_2^2.$$

Proof. By direct calculation. \square

In order to prove Proposition 6.1.2, it suffices to prove the following three equalities for all $a \in \mathbb{Z}$, $b, n \in \mathbb{Z}_{\geq 0}$:

$$\begin{aligned} f_{2,2}(a+1, b+1, n)f_{2,1}(a, b, n) &= f_{2,1}(a+1, b, n)f_{2,2}(a, b, n), \\ f_{2,3}(a+1, b, n)f_{2,2}(a, b, n) &= f_{2,2}(a-2, b-1, n-1)f_{2,3}(a, b, n), \\ f_{2,1}(a-2, b-1, n-1)f_{2,3}(a, b, n) &= f_{2,3}(a+1, b+1, n)f_{2,1}(a, b, n). \end{aligned}$$

This is straightforward by Lemma B.2.1 and the definitions of $f_{2,1}$, $f_{2,2}$, and $f_{2,3}$.

B.3. Proof of Proposition 6.2.7. Let $a_1 \in \mathbb{Z}$, $a_2, b_1, b_2 \in \mathbb{Z}_{\geq 0}$, and let $v \in L(a_1, a_2; b_1, b_2)$ be a highest weight vector. Then, we have

$$h_1 v = [b_1]\{a_1 - b_1 - 1\}v, \quad h_2 v = [b_1 + b_2]\{a_1 + a_2 - (b_1 + b_2) - 1\}v, \quad \tau_2^{-1}(v) = f_2^{(a_2)}v.$$

We need to prove that

$$(18) \quad h_2(v \boxed{0}) = [b_1 + b_2 + 1]\{a_1 + a_2 - (b_1 + b_2)\}v \boxed{0},$$

$$(19) \quad h_2(v \boxed{1}) = [b_1 + b_2]\{a_1 + a_2 - (b_1 + b_2) - 1\}v \boxed{1},$$

$$(20) \quad h_2(v \boxed{-1}) = [b_1 + b_2]\{a_1 + a_2 - (b_1 + b_2) - 1\}v \boxed{-1}.$$

We compute as follows:

$$\begin{aligned} h_2(v \boxed{0}) &= \tau_2(h_1 \cdot \tau_2^{-1}(v \boxed{0})) \\ &= \tau_2(h_1(\tau_2^{-1}(v) \otimes u_0)). \end{aligned}$$

Since $\tau_2^{-1}(v) \in L(a_1, a_2; b_1, b_2)_{a_1+a_2, b_1+b_2, 0}$, we have $\tau_2^{-1}(v) \otimes u_0 \in (L(a_1, a_2; b_1, b_2) \otimes V_2)_{a_1+a_2+2, b_1+b_2, 0}$. This proves equation (18).

Next, we prove equation (19). We have

$$\begin{aligned} h_2(v \boxed{1}) &= \tau_2(h_1 \cdot \tau_2^{-1}(v \boxed{1})), \\ \tau_2^{-1}(v \boxed{1}) &= \tau_2(v) \otimes u_2 - \frac{q^{-b+1}(q - q^{-1})}{\{a - b - 1\}} \tau_2^{-1}(f_1 v) \otimes u_0 - pq^{a-2b} \tau_2^{-1}(v) \otimes u_{-2}. \end{aligned}$$

Since $\tau_2(v) \otimes u_2, \tau_2^{-1}(v) \otimes u_{-2} \in (L(a_1, a_2; b_1, b_2) \otimes V_2)_{a_1+a_2, b_1+b_2, 0}$, it remains to show that $\tau_2^{-1}(f_1 v) \otimes u_0 \in (L(a_1, a_2; b_1, b_2) \otimes V_2)_{a_1+a_2, b_1+b_2, 0}$. Indeed, we have

$$\begin{aligned} h_1 \tau_2^{-1}(f_1 v) &= h_1 f_2^{(a_2+1)} f_1 v \\ &= h_1(f_2 f_1 f_2^{(a_2)} - [a_2] f_1 f_2^{(a_2+1)})v \\ &= h_1 f_2 f_1 f_2^{(a_2)} v \\ &= e_1 f_1 f_2 f_1 f_2^{(a_2)} v \\ &= e_1(f_1^{(2)} f_2 + f_2 f_1^{(2)}) f_2^{(a_2)} v \\ &= f_2 e_1 f_1^{(2)} f_2^{(a_2)} v \\ &= [b_1 + b_2 - 1]\{a_1 + a_2 - (b_1 + b_2) - 2\} f_2 f_1 f_2^{(a_2)} v \\ &= [b_1 + b_2 - 1]\{a_1 + a_2 - (b_1 + b_2) - 2\} \tau_2^{-1}(f_1 v), \end{aligned}$$

as desired. Hence, equation (19) holds.

Finally, we prove equation (20). We have

$$\begin{aligned} h_2(v \boxed{-1}) &= \tau_2(h_1 \cdot \tau_2^{-1}(v \boxed{-1})), \\ \tau_2^{-1}(v \boxed{-1}) &= \tau_2^{-1}(f_1 v) \otimes u_0 - q^{b_1} [b_1] \tau_2^{-1}(v) \otimes u_{-2} - pq^{a_1-b_1-2} [b_1] \tau_2^{-1}(v) \otimes u_2. \end{aligned}$$

Since $\tau_2^{-1}(f_1 v) \otimes u_0, \tau_2^{-1}(v) \otimes u_{-2}$, and $\tau_2^{-1}(v) \otimes u_2$ belong to $(L(a_1, a_2; b_1, b_2) \otimes V_2)_{a_1+a_2, b_1+b_2, 0}$, so does $\tau_2^{-1}(v \boxed{-1})$. This proves equation (20). This completes the proof of the proposition.

B.4. Proof of Lemma 6.2.10. Let x, y , and z be as in the statement of Proposition 6.2.10. Recall that $v \in L = L(a_1, a_2; b_1, b_2)$ is a highest weight vector, and that $h_1 v = [b_1]\{a_1 - b_1 - 1\}v$, $h_2 v = [b_1 + b_2]\{a_1 + a_2 - (b_1 + b_2) - 1\}v$, where $h_1 = [e_1, f_1]_1$, $h_2 = \tau_2(h_1) = [[e_2, e_1]_{-1}, [f_1, f_2]_1]_1$.

Lemma B.4.1. *The following hold.*

$$e_2 f_{2,1} v = x v, \quad e_2 f_{2,2} v = y v, \quad e_2 (f_{2,3} f_1) v = z f_1 v.$$

Proof. Let $\lambda = a_1 \delta_1 + a_2 \delta_2$. Since $L_\lambda = \mathbb{Q}(p, q)v$ and $L_{\lambda - \gamma_1} = \mathbb{Q}(p, q)f_1 v$, there exist $X, Y, Z \in \mathbb{Q}(p, q)$ such that $e_2 f_{2,1} v = Xv$, $e_2 f_{2,2} v = Yv$, and $e_2 (f_{2,3} f_1) v = Zf_1 v$. By the definition of $f_{2,i}$'s, we have $f_2 v = f_{2,1} v + f_{2,2} v$. Applying e_2 to this equation, we obtain

$$(21) \quad [a_2] = X + Y.$$

In addition, since $f_{2,1} v \in L_{a_1+1, b_1+1, 0}$ and $f_{2,2} v \in L_{a_1+1, b_1, 0}$, it follows that

$$\begin{aligned} e_2 e_1 f_1 (f_{2,1} v + f_{2,2} v) &= e_2 ([b_1 + 1]\{a_1 - b_1 - 1\}f_{2,1} v + [b_1]\{a_1 - b_1\}f_{2,2} v) \\ &= ([b_1 + 1]\{a_1 - b_1 - 1\}X + [b_1]\{a_1 - b_1\}Y)v. \end{aligned}$$

Also, we have

$$\begin{aligned} e_2 e_1 f_1 f_2 v &= (h_2 + q^{-1}e_1 e_2 f_1 f_2 + qe_2 e_1 f_2 f_1 - e_1 e_2 f_2 f_1)v \\ &= (h_2 + q^{-1}e_1 f_1 e_2 f_2 + qe_2 f_2 e_1 f_1 - e_1 (f_2 e_2 + \frac{k_2 - k_2^{-1}}{q - q^{-1}})f_1)v \\ &= (h_2 + q^{-1}e_1 f_1 e_2 f_2 + qe_2 f_2 e_1 f_1 - e_1 f_1 \frac{qk_2 - q^{-1}k_2^{-1}}{q - q^{-1}})v \\ &= ([b_1 + b_2]\{a_1 + a_2 - (b_1 + b_2) - 1\} \\ &\quad + (q + q^{-1})[a_2][b_1]\{a_1 - b_1 - 1\} - [a_2 + 1][b_1]\{a_1 - b_1 - 1\})v \\ &= ([b_1 + b_2]\{a_1 + a_2 - (b_1 + b_2) - 1\} + [a_2 - 1][b_1]\{a_1 - b_1 - 1\})v. \end{aligned}$$

Combining these two equalities, we have

$$(22) \quad \begin{aligned} [b_1 + b_2]\{a_1 + a_2 - (b_1 + b_2) - 1\} + [a_2 - 1][b_1]\{a_1 - b_1 - 1\} \\ = [b_1 + 1]\{a_1 - b_1 - 1\}X + [b_1]\{a_1 - b_1\}Y. \end{aligned}$$

Solving the system of equations (21), (22), we obtain $X = x$ and $Y = y$.

Let us determine Z . By the definition of $\mathbf{U}_{<0}^j$, we see that $\dim(\mathbf{U}_{<0}^j)_{-\gamma_1 - \gamma_2} = 3$, and hence, $L_{-\gamma_1 - \gamma_2} = \text{Span}_{\mathbb{Q}(p, q)}\{f_1 f_{2,1} v, f_1 f_{2,2} v, f_{2,3} f_1 v\}$. Therefore, there exist $s, t \in \mathbb{Q}(p, q)$ such that $f_2 f_1 v = s f_1 f_{2,1} v + t f_1 f_{2,2} v + f_{2,3} f_1 v$. applying e_1 to this equation, we deduce that

$$[b_1]\{a_1 - b_1 - 1\}f_2 v = s[b_1 + 1]\{a_1 - b_1 - 1\}f_{2,1} v + t[b_1]\{a_1 - b_1\}f_{2,2} v.$$

Since $f_2 v = f_{2,1} v + f_{2,2} v$, we obtain $s = \frac{[b_1]}{[b_1 + 1]}$ and $t = \frac{\{a_1 - b_1 - 1\}}{\{a_1 - b_1\}}$. In addition, we have

$$[a_2 + 1]f_1 v = e_2 f_2 f_1 v = (sx + ty + Z)f_1 v,$$

and hence,

$$Z = [a_2 + 1] - sx - ty = z,$$

as desired. □

Now, we can complete the proof of Lemma 6.2.10 by direct calculation using the previous lemma; we omit the details.

B.5. Proof of Proposition 6.2.11. Since we have assumed that $(\mathcal{L}(a_1, a_2; b_1, b_2), \mathcal{B}(a_1, a_2; b_1, b_2))$ is a j -crystal basis of $L(a_1, a_2; b_1, b_2)$, we may identify $v + q\mathcal{L}(a_1, a_2; b_1, b_2) \in \mathcal{B}(a_1, a_2; b_1, b_2)$ with the semistandard double tableau

$$\left(\begin{array}{|c|c|c|} \hline 0 & \cdots & 0 \\ \hline -1 & \cdots & -1 \\ \hline \end{array} \begin{array}{|c|c|c|} \hline 0 & \cdots & 0 \\ \hline \end{array} \begin{array}{|c|c|c|} \hline 1 & \cdots & 1 \\ \hline \end{array} \right).$$

$\underbrace{\hspace{1.5cm}}_{b_2} \quad \underbrace{\hspace{1.5cm}}_{b_1} \quad \underbrace{\hspace{1.5cm}}_{a_1-b_1}$

Recall that $M := L_{\lambda+2\delta_1-\gamma_1-\gamma_2} \cap \text{Ker}(e_1) \cap \text{Ker}(e_2) \cap \text{Ker}(h_1 - [b_1]\{a_1 - b_1 - 1\})$ is spanned by m_1 and m_2 . Since $h_2 m = \tau_2(h_1 \tau_2^{-1}(m))$ for $m \in M$, if $m \in M$ is an h_2 -eigenvector, then $\tau_2^{-1}(m)$ is an h_1 -eigenvector. So, let us consider the vector space $\tau_2^{-1}(M) = \tilde{f}_2^{a_2-1}(M)$. Since $m_1 + q\mathcal{L} = v \otimes u_2 + q\mathcal{L}$, we have $\tilde{f}_2^{a_2-1}(m_1) + q\mathcal{L} = \tilde{f}_2^{a_2-1}(v) \otimes u_2 + q\mathcal{L}$. Also, since $m_2 + q\mathcal{L} = \tilde{f}_1 \tilde{f}_2(v) \otimes u_0 + q\mathcal{L}$, we have $\tilde{f}_2^{a_2-1}(m_2) + \mathcal{L} = \tilde{f}_2^{a_2-1} \tilde{f}_1 \tilde{f}_2(v) \otimes u_0 + q\mathcal{L}$. By identifying $v + q\mathcal{L}(a_1, a_2; b_1, b_2)$ with the semistandard double tableau above, it is easy to see that $\tilde{f}_2^{a_2-1} \tilde{f}_1 \tilde{f}_2(v) + q\mathcal{L} = \tilde{f}_1 \tilde{f}_2^{a_2}(v) + q\mathcal{L}$. Indeed, $\tilde{f}_2^{a_2-1}(v) + q\mathcal{L}(a_1, a_2; b_1, b_2)$ and $\tilde{f}_1 \tilde{f}_2^{a_2}(v) + q\mathcal{L}(a_1, a_2; b_1, b_2)$ are identified with

$$\left(\begin{array}{|c|c|c|} \hline 0 & \cdots & 0 \\ \hline -2 & \cdots & -2 \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline 0 & \cdots & 0 & 0 \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & 2 \\ \hline \end{array} \right),$$

$$\left(\begin{array}{|c|c|c|} \hline 0 & \cdots & 0 \\ \hline -2 & \cdots & -2 \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline 0 & \cdots & 0 & -1 \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline 2 & 2 & \cdots & 2 \\ \hline \end{array} \right),$$

respectively. Hence, $\tilde{f}_2^{a_2-1}(M)$ is spanned by an element in $\tilde{f}_2^{a_2-1}(v) \otimes u_2 + q\mathcal{L}$ and an element in $\tilde{f}_1 \tilde{f}_2^{a_2}(v) \otimes u_0 + q\mathcal{L}$. By the representation theory of \mathbf{U}_1^j , this implies that $\tilde{f}_2^{a_2-1}(M)$ is spanned by an h_1 -eigenvector in $\tilde{f}_2^{a_2}(v) \otimes u_2 + q\mathcal{L}$ with eigenvalue $[b_1 + b_2]\{a_1 + (a_2 - 1) - (b_1 + b_2) - 1\}$ and an h_1 -eigenvector in $\tilde{f}_1 \tilde{f}_2^{a_2}(v) \otimes u_0 + q\mathcal{L}$ with eigenvalue $[b_1 + b_2 - 1]\{a_1 + (a_2 - 1) - (b_1 + b_2 - 1) - 1\}$. Applying $\tilde{e}_2^{a_2-1}$ on these vectors, we conclude that there exist an h_2 -eigenvector $v_2 \in M$ with eigenvalue $[b_1 + b_2]\{a_1 + (a_2 - 1) - (b_1 + b_2) - 1\}$ such that $v_2 \in v \otimes u_2 + q\mathcal{L}$ and an h_2 -eigenvector $v_{-2} \in M$ with eigenvalue $[b_1 + b_2 - 1]\{a_1 + (a_2 - 1) - (b_1 + b_2 - 1) - 1\}$ such that $v_{-2} \in \tilde{f}_1 \tilde{f}_2(v) \otimes u_0$. This proves the proposition.

B.6. Proof of Proposition 9.2.1. We prove the assertion by induction on $r \geq 2$. The case $r = 2$ is already proved by Proposition 6.2.11. Let $r \geq 3$ and assume that the assertion holds for $r' < r$. Then, there exist $v_s \in \mathcal{L}$ satisfying the assertion for $s \in S \setminus \{\pm r\}$. We set $L(r-1) := \bigoplus_{s \in S \setminus \{\pm r\}} \mathbf{U}^j v_s$, $\mathcal{L}(r-1) := \mathcal{L} \cap L(r-1)$, and $\mathcal{B}(r-1) := \mathcal{B} \cap (\mathcal{L}(r-1)/q\mathcal{L}(r-1))$. Let $\lambda \in \Lambda^j$ denote the weight of v , and set $\mu := \lambda + 2\delta_1 - \sum_{n=1}^r \gamma_n$. It is straightforward to verify that $\mathcal{B}_\nu \cap \mathcal{B}(r-1) = \mathcal{B}_\nu$ for all $\nu > \mu$, and that

$$\mathcal{B}_\mu \cap \mathcal{B}(r-1) = \begin{cases} \mathcal{B}_\mu \setminus \{v \otimes u_r + q\mathcal{L}, \tilde{f}_1 \cdots \tilde{f}_r(v) \otimes u_0 + q\mathcal{L}\} & \text{if } \pm r \in S, \\ \mathcal{B}_\mu \setminus \{v \otimes u_r + q\mathcal{L}\} & \text{if } r \in S \text{ and } -r \notin S, \\ \mathcal{B}_\mu \setminus \{\tilde{f}_1 \cdots \tilde{f}_r(v) \otimes u_0 + q\mathcal{L}\} & \text{if } r \notin S \text{ and } -r \in S, \\ \mathcal{B}_\mu & \text{if } \pm r \notin S. \end{cases}$$

This implies that the quotient module $L/L(r-1)$ has no weights greater than μ , and the weight space $(L/L(r-1))_\mu$ is spanned by $v \otimes u_r + L(r-1)$ and $\tilde{f}_1 \cdots \tilde{f}_r(v) \otimes u_0 + L(r-1)$. By the representation theory of \mathbf{U}_{r-1}^j , the space $(L/L(r-1))_\mu$ is spanned by at most two \mathbf{U}_{r-1}^j -highest weight vectors. In the same manner as in the proof of Proposition 6.2.11, we see that there exist h_r -eigenvectors $v'_r \in (L/L(r-1))_\mu$ such that $v'_r \in v \otimes u_r + q(\mathcal{L}/\mathcal{L}(r-1))$ if $r \in S$, and $v'_{-r} \in (L/L(r-1))_\mu$ such that $v'_r \in \tilde{f}_1 \cdots \tilde{f}_r(v) \otimes u_0 + q(\mathcal{L}/\mathcal{L}(r-1))$ if $-r \in S$. Therefore, we obtain a \mathbf{U}^j -highest weight vector $v_r \in L_\mu$ such that $v_r + L(r-1) = v'_r$ if $r \in S$, and $v_{-r} \in L_\mu$ such that $v_{-r} + L(r-1) = v'_{-r}$ if $-r \in S$.

Next, we show that $v_{\pm r} \in \mathcal{L}$. If $v_r \notin \mathcal{L}$, then there exists $(k, l) \in (\mathbb{Z}_{>0} \times \mathbb{Z}) \cup (\{0\} \times \mathbb{Z}_{>0})$ such that $p^k q^l v_r \in \mathcal{L} \setminus q\mathcal{L}$. Since $v_r + L(r-1) \in v \otimes u_r + q\mathbf{A}_0 v \otimes u_r + q\mathbf{A}_0 \tilde{f}_1 \cdots \tilde{f}_r(v) \otimes u_r + L(r-1)$, the vector $p^k q^l v_r + q\mathcal{L}$ is a linear combination of elements in $\mathcal{B}_\mu \setminus \{v \otimes u_r + q\mathcal{L}, \tilde{f}_1 \cdots \tilde{f}_r(v) \otimes u_0 + q\mathcal{L}\}$, and hence, $\tilde{e}_i(p^k q^l v_r + q\mathcal{L}) \neq 0$ for some $i \in \overline{\mathbb{I}}^j$. However, since $p^k q^l v_r$ is a \mathbf{U}_r^j -highest weight vector, we have $\tilde{e}_i(p^k q^l v_r + q\mathcal{L}) = 0$ for all $i \in \overline{\mathbb{I}}^j$. This causes a contradiction. Thus, we obtain $v_r \in \mathcal{L}$. Similarly, we can prove that $v_{-r} \in \mathcal{L}$.

It remains to show that $\mathbf{U}^j v_{\pm r} \simeq L(\alpha^{\pm r}; \beta^{\pm r})$. This is done by determining the eigenvalues of $v_{\pm r}$ for k_i and h_i , $i \in \mathbb{I}^j$. To do this, we identify $v + q\mathcal{L}$ with the semistandard double Young tableau $(T_{\alpha'}; T_{\beta'})$, where $\alpha' = (\alpha_1, \dots, \alpha_r)$ and $\beta' = (\beta_1, \dots, \beta_{r-1})$. By the representation theory of \mathbf{U}_{r-1}^j , the h_i -eigenvalue of $v_{\pm r}$ for $i \in \mathbb{I}^j \setminus \{r\}$ are determined by regarding $v_{\pm r} + q\mathcal{L}$ as an element of $\mathcal{B}_r^{\otimes N+1}$ and then ignoring all the $\pm r$'s. For example, since $v_r + q\mathcal{L} = v \otimes u_r + q\mathcal{L} = (\text{EM}(T_{\alpha'}), \text{ME}(T_{\beta'}), r)$, the k_i - and h_i -eigenvalues of v_r , $i \neq r$, are the same as those of v . Similarly, $s_r(v_{\pm r} + q\mathcal{L})$ determines the k_r - and h_r -eigenvalues. This proves the proposition.

REFERENCES

- [BW13] H. Bao and W. Wang, A new approach to Kazhdan-Lusztig theory of type B via quantum symmetric pairs, arXiv:1310.0103v2.
- [BWW16] H. Bao, W. Wang, and H. Watanabe, Multiparameter quantum Schur duality of type B , arXiv:1609.01766. To appear in Proc. Amer. Math. Soc.
- [DJ92] R. Dipper and G. James, Representations of Hecke algebras of type B_n , J. Algebra 146 (1992), no. 2, 454-481.
- [D93] M. J. Dyer, Hecke algebras and shellings of Bruhat intervals, Compositio Math. 89 (1993), no. 1, 91-115.
- [GAP16] The GAP group, GAP - Groups, algorithms, and programming, Version 4.8.3, 2016.
- [G86] A. Gyoja, A q -analogue of Young symmetrizer, Osaka J. Math. 23 (1986), no. 4, 841-852.
- [H74] P. N. Hoefsmit, Representations of Hecke algebras of finite groups with BN-pairs of classical type, Thesis (Ph.D.)-The University of British Columbia (Canada). ProQuest LLC, Ann Arbor, MI, 1974.
- [HK02] J. Hong and S. J. Kang, Introduction to Quantum Groups and Crystal Bases, Graduate Studies in Mathematics Vol. 42., American Mathematical Society, Providence, RI, 2002. xviii+307 pp.
- [J86] M. Jimbo, A q -analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang-Baxter equation, Lett. Math. Phys. 11 (1986), no. 3, 247-252.
- [Ka91] M. Kashiwara, On crystal bases of the q -analogue of universal enveloping algebras, Duke Math. J. 63 (1991), no. 2, 465-516.
- [Ko14] S. Kolb, Quantum symmetric Kac-Moody pairs, Adv. Math. 267 (2014), 395-469.
- [KP11] S. Kolb and J. Pellegrini, Braid group actions on coideal subalgebras of quantized enveloping algebras, J. Algebra 336 (2011), 395-416.
- [Kw09] J. H. Kwon, Crystal graphs and the combinatorics of Young tableaux, Handbook of Algebra Vol. 6, 473-504, Handb. Algebr., 6, Elsevier/North-Holland, Amsterdam, 2009.
- [Le99] G. Letzter, Symmetric pairs for quantized enveloping algebras, J. Algebra 220 (1999), no. 2, 729-767.
- [LS91] S. Z. Levendorskii and Y. S. Soibelman, The quantum Weyl group and a multiplicative formula for the R -matrix of a simple Lie algebra, Funct. Anal. Appl. 25 (1991), no. 2, 143-145.
- [Lu90] G. Lusztig, Quantum groups at roots of 1, Geom. Dedicata, 33 (1990), 89-113.
- [Lu94] G. Lusztig, Introduction to Quantum Groups, Reprint of the 1994 edition. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2010. xiv+346 pp.

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